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On Projective Planes Of Order Nine

By Marshall Hall, Jr., J. Dean Swift and Raymond Killgrove

1. Introduction. There are four known projective planes of order nine: the desarguesian plane; the plane coordinatizable by a Veblen-Wedderburn system which is not a field; its dual; and a self-dual non-desarguesian plane. The last was originally found by Veblen and Wedderburn [4]; its structure has been generalized by Hughes [3].

Each of these planes was known to have an elementary abelian addition in an appropriate ternary ring. A natural question was whether or not there were any other planes with this property. A search was made partly by hand, partly by machine for such planes. The result of the search is that no further planes exist with this property but that the three non-desarguesian planes all may be coordinatized in more than one way with an elementary abelian addition. For notation and terminology used in this paper the reader is referred to [1].

2. The additive pencil and lines consistent with it. In a plane of order 9 with an elementary abelian addition we may assume initially three pencils of finite lines $x = c$, $y = c$, and $y = x + c$, $c = 0, 1, \cdots, 8$, where the addition in the last pencil is that of the elementary abelian group. Specifically this last pencil may be represented by the latin square:

$$
\begin{array}{cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 & 1 & 2 & 0 & 4 & 5 & 3 & 7 & 8 & 6 \\
 & 2 & 0 & 1 & 5 & 3 & 4 & 8 & 6 & 7 \\
 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 \\
(2.1) & 4 & 5 & 3 & 7 & 8 & 6 & 1 & 2 & 0 \\
 & 5 & 3 & 4 & 8 & 6 & 7 & 2 & 0 & 1 \\
 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 \\
 & 7 & 8 & 6 & 1 & 2 & 0 & 4 & 5 & 3 \\
 & 8 & 6 & 7 & 2 & 0 & 1 & 5 & 3 & 4
\end{array}
$$

where, as is customary, in each row of the square: $y_0y_1y_2y_3y_4y_5y_6y_7y_8$ is to be read as the second row of the permutation:

$$
(2.2) \qquad \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \end{pmatrix}
$$

and represents a line whose nine finite points are (i, y_i) $i = 0, 1, \cdots, 8$.

In our search it is sufficient to find the affine planes of order nine; the extension to projective planes by adding the points at infinity is trivial.

It turns out that there are 2241 lines consistent with the pencil (2.1), these being represented by permutations (2.2) which do not have as many as two points (i, y_i) in common with any one of the lines of (2.1). In practice the lines for which $y_0 = 0$

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are first found. These are then subjected to the permutations given by (2.1) to obtain the full list. If L is a consistent line and A_α is a permutation of (2.1) then LA_α is also and conversely. Indeed if L_1 and L_2 are any two lines and P any permutation, the number of intersections (coincidences) of L_1 and L_2 is obviously equal to the number for L_1P and L_2P. In our case the group goes into itself under multiplication by A_α so that the consistency is maintained.

The discovery of the 249 lines through (0, 0) may be done by hand or by simple mechanical sorting procedures. To check and to facilitate the consideration of other possible structures a program for the automatic computer *SWAC* has been written which will determine the lines consistent with any square which is the table of a loop.

The problem of the search is now to find sets of 63 additional lines from the 2241 such that each resulting total of 90 lines contains exactly one line joining any two distinct points, (a, b) and (c, d). For $a = c$ or $b = d$ we have the line $x = a = c$ or $y = b = d$ in one of the pre-determined pencils.

The search described would be far too extensive in practice and a procedure must be found to reduce the total number of sets to be looked for and the total number of isomorphic planes to be found. To do this we consider automorphisms which preserve the 27 fixed lines of the three basic pencils. The group G which takes the three pencils into themselves consists of $81 \cdot 48$ automorphisms. Of these 81 translations T have the form:

$$(x, y) \to (x + a, y + b). \tag{2.3}$$

The subgroup H fixing (0, 0) has 48 elements and is essentially the automorphism group of the elementary abelian group of order 9. These automorphisms are of the form:

$$(x, y) \to (\alpha(x), \alpha(y)) \tag{2.4}$$

where α is a permutation of $0, 1, \cdots, 8$ which leaves 0 fixed. This group contains a subgroup U of order 6. The permutations α of U are:

$$\begin{aligned} \alpha_1 &= I \\ \alpha_2 &= (345)(687) \\ \alpha_3 &= (354)(678) \\ \alpha_4 &= (36)(47)(58) \\ \alpha_5 &= (37)(48)(56) \\ \alpha_6 &= (38)(46)(57) \end{aligned} \tag{2.5}$$

The group H may now be given by

$$H = U + U\alpha_7 + U\alpha_{13} + U\alpha_{19} + U\alpha_{25} + U\alpha_{31} + U\alpha_{37} + U\alpha_{43} \tag{2.6}$$

where

$$\begin{aligned} \alpha_7 &= (12)(36)(48)(57) \\ \alpha_{13} &= (13472685) \\ \alpha_{19} &= (1428)(3765) \\ \alpha_{25} &= (15862743) \\ \alpha_{31} &= (16452387) \\ \alpha_{37} &= (17832546) \\ \alpha_{43} &= (1824)(3567). \end{aligned} \tag{2.7}$$

To apply the automorphisms T of 2.3 to a line L the product $A_a^{-1}LA_b$ is computed where A_a and A_b are the permutations of (2.1) containing taking 0 to a and b respectively. To apply an automorphism α of H to L, $\alpha^{-1}L\alpha$ is calculated. As these automorphisms leave the lines (2.1) invariant they take consistent lines to consistent lines and this fact may be used in the construction and checking of the consistent lines. The discussion of the method by which the 2241 lines are obtained from 249 may be simply phrased in terms of translations.

There are still other automorphisms which exchange the pencils $x = c$ (through (∞) on L_∞), $y = c$ (through (0) on L_∞) and $y = x + c$ (through (1) on L_∞) among themselves. The reflection R about $y = x$ is given by

$$(x, y) = (y, x). \tag{2.10}$$

This is applied to L by computing L^{-1}.

There is also the automorphism C:

$$(x, y) \to (x, x - y). \tag{2.11}$$

C and R together generate a group K of order 6 which permutes (∞), (0) and (1) in all possible ways.

3. Classification of consistent lines. The 249 lines which pass through (0, 0) and are consistent with (2.1) are interchanged by the group H. These lines fall into 12 equivalence classes with respect to this group.

We list here one representative of each class and the number of lines in each class.

(3.1)

Class	Line										Number of Conjugates
1	0	2	1	6	8	7	3	5	4		1
2	0	2	1	6	8	7	4	3	5		16
3	0	2	1	7	6	8	4	3	5		8
4	0	2	1	7	6	8	5	4	3		8
5	0	2	3	1	6	8	5	4	7		48
6	0	2	3	8	7	1	4	6	5		48
7	0	3	5	2	1	6	4	8	7		48
8	0	3	6	2	5	8	1	4	7		6
9	0	3	6	2	7	4	1	8	5		6
10	0	3	6	4	1	8	5	2	7		48
11	0	3	6	4	7	1	8	2	5		6
12	0	3	6	7	1	4	5	8	2		6
										Total	249

The distribution of these lines may be represented as follows:

(3.2)

Form of line	Class											
	1	2	3	4	5	6	7	8	9	10	11	12
$02\cdots$	1	10	2	2	6	6	0	0	0	0	0	0
$0i\cdots(i = 3, \cdots, 8)$	0	1	1	1	7	7	8	1	1	8	1	1

The classes tabulated in (3.1) and (3.2) are equivalences under H alone. With respect to G, obviously each of the 2241 lines is equivalent to one of the lines of

(3.1) but some of the classes may be further amalgamated. For example, let L_2 be the representative of class 2 in (3.1); if we apply to L_2 first the translation $(x, y) \to (x + 3, y + 8)$ and then the automorphism $\alpha_7 = (12)(36)(48)(57)$ of H, the representative of class 3 results:

$$A_3^{-1}L_2A_3 = A_6L_2A_3 = (063)(174)(285)\cdot(12)(364857)(084)(165)(273)$$
$$= (12)(38)(47)(56)$$

$$(12)(36)(48)(57)(12)(38)(47)(56)(12)(36)(48)(57) = (12)(37)(46)(58) = L_2.$$

In a similar manner the classes 5, 6, 7 amalgamate under G. Specifically if we denote the elements of H: (12)(34)(68) and (12)(274658), by α_{12} and α_9 respectively and follow the rule of denoting the line (permutation) representing the ith class of (3.1) by L_i we have:

$$\alpha_{12}A_2^{-1}L_6A_1\alpha_{12} = L_6$$

and

$$\alpha_9^{-1}A_1^{-1}L_5A_6\alpha_9 = L_7\,.$$

Further, classes 9 and 10 amalgamate under G but classes 1, 4, 8, 11, 12 remain distinct. However, using the group K which permutes the basic pencils, we find that classes 8 and 12 are equivalent under the transformation C while 11 and 12 are equivalent under R. Thus classes 8, 11, 12 amalgamate under the group K. These are the only new equivalences provided by K and there finally remain 6 classes: 1; (2, 3); 4; (5, 6, 7); (8, 11, 12); (9, 10).

Although many of the procedures discussed in this section are essentially mechanical in nature, programming is at least as difficult as the hand computations and the latter method was utilized for all these results.

4. Construction of pencils through (0, 0). A plane π containing the additive pencil (2.1) must contain a line through (0, 0) and (1, 2), that is a line whose representation is of the form $02\cdots$. From (3.2) it is clear that such a line must belong to one of the classes 1–6. But, using the equivalences deduced under G, we may assume that it belongs to one of the classes 1, 3, 4, 6 since a line of class 2 is equivalent to one of class 3 and a line of class 5 will be transformed to one of class 6.

Among these four classes we shall establish a preference order or hierarchy in the order 1, 6, 4, 3. That is, we shall eliminate all lines of class 1 from consideration when making a search using a class 6 line; again when using a class 4 line we shall eliminate any line of class 1, 5, 6, or 7; finally using a class 3 line for a start any line of classes, 1, 4, 5, 6, and 7 is excluded. This preference order helps to eliminate duplications since each plane will appear only with its most preferred line.

The first stage of the search consists in finding, for each start, the possible pencils through (0, 0). When A_0 of (2.1) is eliminated and we ignore the first column, such a pencil is represented by a latin square of order 8 on the numbers $1, 2, \cdots, 8$ whose first line is the designated start and whose subsequent lines are selected from the 249 permissible lines through (0, 0) with due respect for preference order.

A considerable portion of this search was carried out by hand. The complete

search was made independently by SWAC and the results compared for the duplicated parts.

The mechanical procedure first finds for each of the four starts, the lines through $(0, 0)$, $(1, i), i = 3, \cdots, 8$ consistent with the start; a further rejection is then made on the basis of preference. Next, a particular line through (0, 0), (1, 3) is selected and all remaining lines consistent with it are used in the high speed memory of SWAC to form by a continuation of these processes all possible latin squares. Then the next line through (1, 3) is taken and so on to completion.

With the set of possible pencils before us, we may proceed in either of two ways (or a combination of them). We may eliminate equivalent pencils on the basis of the automorphism groups H and K which leave (0, 0) fixed, and attempt to complete the surviving representatives to planes or we may eliminate the pencils which cannot be completed and then discuss possible equivalences among the survivors. The first choice is better adapted to hand methods, the second to machines.

In the cases which produced a large number of possible pencils it was desirable to introduce a secondary preference order. In particular, the class 1 line, L_1, goes into itself under all substitutions of H; moreover lines through (0, 0) in classes 2, 3, 4, 5 and 6 are not consistent with L_1. Hence any line through (0, 0) consistent with L_1 may be carried by elements of H into one of L_7, L_{10}, L_8, L_9, L_{11}, L_{12}. These, in this order, were adopted as secondary preferences and the total output of squares of type 1 was thus reduced to 50.

Under the automorphisms of $H \cdot K$ these pencils fall into 17 classes. Similarly, the class 6 start produced 57 pencils ln 11 classes; the class 4 start 12 pencils in 3 classes. No pencils are found in class 3.

In the mechanical procedure, the 249 possible lines through a point $(x, 0)$ are first found by translating the 249 lines through (0, 0). Each of these lines is then compared with a square and those lines which intersect no line of the square in more than one point are stored and eventually tabulated with the square. It frequently occurred that in the tabulation for a particular square no line through $(0, y)$ was present for at least one y. This square could be at once rejected as impossible to complete.

In all routines used on SWAC the major operation was one of sorting. To facilitate this sorting, positional coding was used. Each element y_i of a line was assigned eight consecutive spaces in a register (since *SWAC* uses 36 bit words, double precision was required to store a full line). The number i is represented by a 1 in the ith space and 0's in the other 7; 0 is represented by 0's in all eight places. Using complementation and extraction, the number of intersections was representable as the number of 1 bits in a resulting word pair.

5. Representative pencils which are completable to a plane. Most of the squares found cannot be completed; further, most of the squares which do lead to planes do so uniquely. After a few additional pencils are added from the tabulations described above the remainder of the lines are essentially decided and it is a simple matter to write them down.

When, in either order, the equivalences of pencils through (0, 0) have been taken into account and completability determined, we find 12 pencils remaining, 8 of class

1, 1 of class 6 and 3 of class 4. These are tabulated here omitting the lines $x = 0$, $y = 0$, $x = y$ common to all cases. The numbering corresponds to the original position of the square in the original list of squares mentioned above.

Class 1 representative pencils which can be completed

(1.4)

0	2	1	6	8	7	3	5	4
0	3	5	2	1	6	4	8	7
0	4	3	8	2	1	7	6	5
0	5	4	1	7	2	8	3	6
0	6	8	7	5	4	2	1	3
0	7	6	4	3	8	5	2	1
0	8	7	5	6	3	1	4	2

(1.27)

0	2	1	6	8	7	3	5	4
0	3	6	4	1	8	5	2	7
0	4	7	1	3	6	2	8	5
0	5	8	2	7	3	4	6	1
0	6	4	7	5	2	8	1	3
0	7	5	8	2	4	1	3	6
0	8	3	5	6	1	7	4	2

(1.26)

0	2	1	6	8	7	3	5	4
0	3	6	4	1	8	5	2	7
0	4	7	1	3	6	2	8	5
0	5	8	2	6	4	7	1	3
0	6	5	7	2	3	8	4	1
0	7	3	8	5	1	4	6	2
0	8	4	5	7	2	1	3	6

(1.28)

0	2	1	6	8	7	3	5	4
0	3	6	4	1	8	5	2	7
0	4	7	5	6	2	1	8	3
0	5	8	1	7	4	2	3	6
0	6	5	7	2	3	8	4	1
0	7	3	8	5	1	4	6	2
0	8	4	2	3	6	7	1	5

(1.30)

0	2	1	6	8	7	3	5	4
0	3	6	4	1	8	5	2	7
0	4	7	8	2	3	1	6	5
0	5	8	1	7	4	2	3	6
0	6	4	7	5	2	8	1	3
0	7	5	2	3	6	4	8	1
0	8	3	5	6	1	7	4	2

(1.36)

0	2	1	6	8	7	3	5	4
0	3	6	2	5	8	1	4	7
0	4	8	7	2	3	5	6	1
0	5	7	4	6	2	8	1	3
0	-6	3	1	7	4	2	8	5
0	7	5	8	3	1	4	2	6
0	8	4	5	1	6	7	3	2

(1.35)

0	2	1	6	8	7	3	5	4
0	3	6	2	5	8	1	4	7
0	4	8	5	6	1	7	2	3
0	5	7	8	1	3	4	6	2
0	6	3	1	7	4	2	8	5
0	7	5	4	2	6	8	3	1
0	8	4	7	3	2	5	1	6

(1.44)

0	2	1	6	8	7	3	5	4
0	3	6	2	7	4	1	8	5
0	4	8	5	2	6	7	3	1
0	5	7	8	3	2	4	1	6
0	6	3	1	5	8	2	4	7
0	7	5	4	6	1	8	2	3
0	8	4	7	1	3	5	6	2

Class 6 representative pencil which can be completed

(6.17)

0	2	3	8	7	1	4	6	5
0	3	6	4	8	2	5	1	7
0	4	7	5	1	6	2	8	3
0	5	8	1	6	4	7	3	2
0	6	5	7	2	3	8	4	1
0	7	4	2	3	8	1	5	6
0	8	1	6	5	7	3	2	4

Class 4 representative pencils which can be completed

(4.3)

0	2	1	7	6	8	5	4	3
0	3	6	2	7	4	1	8	5
0	4	8	5	2	6	7	3	1
0	5	7	8	3	2	4	1	6
0	6	3	4	1	7	8	5	2
0	7	5	1	8	3	2	6	4
0	8	4	6	5	1	3	2	7

(4.4)

0	2	1	7	6	8	5	4	3
0	3	6	4	7	1	8	2	5
0	4	8	2	3	7	1	5	6
0	5	7	6	2	4	3	8	1
0	6	3	8	5	2	4	1	7
0	7	5	1	8	3	2	6	4
0	8	4	5	1	6	7	3	2

(4.5)

0	2	1	7	6	8	5	4	3
0	3	6	5	8	2	7	1	4
0	4	7	1	3	6	2	8	5
0	5	8	2	7	3	4	6	1
0	6	3	4	1	7	8	5	2
0	7	5	8	2	4	1	3	6
0	8	4	6	5	1	3	2	7

Each of the 12 has a unique completion to a full plane except 1.44 which has nine completions. But, using our preference order, the class 6 plane may be removed from further consideration. In the completion of the 6.17 square the class 1 line, 3 5 4 0 2 1 6 8 7, occurs unequivocally.

The tabulations yield 8 possible pencils through (2, 0) consistent with 1.44. With one exception each of these is uniquely completable; the first has two completions. These pencils are listed here, omitting the lines $x = 2$, $y = 0$ and $y = x + 1$.

(2, 0) pencils for 1.44

(1.44.1)

2	1	0	8	7	6	5	4	3
3	6	0	7	4	2	8	5	1
4	8	0	2	6	5	3	1	7
5	7	0	3	2	8	1	6	4
6	3	0	5	8	1	4	7	2
7	5	0	6	1	4	2	3	8
8	4	0	1	3	7	6	2	5

(1.44.2)

2	1	0	8	7	6	5	4	3
3	6	0	5	1	7	4	2	8
4	8	0	6	3	1	2	7	5
5	7	0	1	8	4	6	3	2
6	3	0	7	2	5	8	1	4
7	5	0	2	4	8	3	6	1
8	4	0	3	6	2	1	5	7

(1.44.3)

2	1	0	8	7	6	5	4	3
3	7	0	2	8	5	4	6	1
4	6	0	3	2	7	8	1	5
5	8	0	6	4	2	1	3	7
6	4	0	7	3	1	2	5	8
7	3	0	5	1	8	6	2	4
8	5	0	1	6	4	3	7	2

(1.44.5)

2	1	0	8	7	6	5	4	3
3	6	0	5	1	7	4	2	8
4	7	0	2	3	8	6	5	1
5	8	0	6	4	2	1	3	7
6	3	0	7	2	5	8	1	4
7	4	0	3	8	1	2	6	5
8	5	0	1	6	4	3	7	2

(1.44.6)

2	1	0	7	6	8	4	3	5
3	7	0	8	2	5	1	6	4
4	6	0	3	7	2	8	5	1
5	8	0	2	4	6	3	1	7
6	4	0	1	3	7	5	2	8
7	3	0	5	8	1	6	4	2
8	5	0	6	1	4	2	7	3

(1.44.7)

2	1	0	8	7	6	5	4	3
3	7	0	2	8	5	4	6	1
4	8	0	6	3	1	2	7	5
5	6	0	7	2	4	3	1	8
6	5	0	1	4	7	8	3	2
7	3	0	5	1	8	6	2	4
8	4	0	3	6	2	1	5	7

(2, 0) pencils for 1.44—*continued*

(1.44.4)

2	1	0	8	7	6	5	4	3
3	8	0	5	6	2	1	7	4
4	7	0	2	3	8	6	5	1
5	6	0	7	2	4	3	1	8
6	5	0	1	4	7	8	3	2
7	4	0	3	8	1	2	6	5
8	3	0	6	1	5	4	2	7

(1.44.8)

2	1	0	8	7	6	5	4	3
3	8	0	5	6	2	1	7	4
4	6	0	3	2	7	8	1	5
5	7	0	1	8	4	6	3	2
6	4	0	7	3	1	2	5	8
7	5	0	2	4	8	3	6	1
8	3	0	6	1	5	4	2	7

To give a final separation of the completable planes it suffices to indicate the pencils through (3, 0) for 1.44.1. Again lines $x = 3$, $y = 0$, $y = x + 6$ are omitted:

(3, 0) pencils for 1.44.1

(1.44.1.1)

1	4	7	0	8	5	2	6	3
2	8	5	0	4	7	1	3	6
3	5	4	0	2	1	6	8	7
4	2	6	0	3	8	5	7	1
5	6	1	0	7	3	4	2	8
7	3	2	0	5	6	8	1	4
8	1	3	0	6	4	7	5	2

(1.44.1.2)

1	8	5	0	3	6	2	7	4
2	4	7	0	6	3	1	5	8
3	5	4	0	2	1	6	8	7
4	1	6	0	5	7	8	3	2
5	6	2	0	8	4	7	1	3
7	3	1	0	4	8	5	2	6
8	2	3	0	7	5	4	6	1

6. Equivalence of completed planes. It remains to recognize the distinct planes among the 19 completed planes (excluding that of class 6). For the purposes of tabulation we distinguish the four previously known planes as follows:

A. The desarguesian plane—coordinatizable by GF(9); this appears as 1.35, this pencil through (0, 0) being readily recognizable as giving the multiplication table of the field.

B. The Veblen-Wedderburn plane—coordinatizable by a near field. Again this multiplication may be recognized in the square given for 1.36.

C. The dual Veblen-Wedderburn plane—This may be exhibited as the plane constructable from 1.44.2.

D. The plane whose coordinatization in terms of an elementary abelian addition was indicated by Hughes [3]. This appears in 1.44.1.1.

Of these, planes A, B, C have lines of the form $y = xm + b$ in their standard form and, using the additive pencil (2.1) and the multiplicative pencil through (0, 0) tabulated in the last section, the remaining lines may easily be added by the reader who wishes a complete tabulation. For B the distributive law $(x_1 + x_2)m = x_1m + x_2m$ holds; the multiplications are automorphisms of the additive group. For C the distributive law $x(m_1 + m_2) = xm_1 + xm_2$ is valid.

For D we give the complete tabulation of 1.44.1.1 below. In order that the reader may follow the details of the arguments on equivalence of planes we also list the completions of 1.44.1.2, 1.44.6, 1.4 and 4.3.

We now maintain that the nineteen planes may be classified as follows:

A. 1.35

B. 1.36, 1.4, 1.44.6, 4.4

C. 1.44.2, 1.44.1.2, 1.44.3, 1.44.4, 1.27, 1.28

D. 1.44.1.1, 1.44.5, 1.44.7, 1.44.8, 1.26, 1.30, 4.3, 4.5.

To begin with class B, 4.4 is the Veblen-Wedderburn plane coordinatized by the system in which -1 is not in the center. This system is designated by the letter

U in [1]; as shown in this reference (p. 274) the same plane results as that produced by the nearfield. The translation $L \to A_1^{-1}LA_2$ or $(x, y) \to (x + 1, y + 2)$ takes 1.4 into 1.44.6. It is somewhat more difficult to show the equivalence of 1.44.6 and 1.36. In the projective completion of 1.44.6 we make the following replacements.

$$(0, 0) \to (\infty)$$

$$(\infty) \to (0, 0)$$

$$(0) \to (0)$$

$$(1, 1) \to (1, 1)$$

$$(\infty)\begin{pmatrix}1&1&1&1&1&1&1&1&1\\0&1&2&3&4&5&6&7&8\end{pmatrix} \to \begin{pmatrix}0\\0\end{pmatrix}(1)\begin{pmatrix}1&2&3&4&5&6&7&8\\1&2&3&4&5&6&7&8\end{pmatrix}$$

where the last replacement indicates that the line $x = 1$ is replaced by the line $x = y$ in the manner shown. With the new coordinates 1.44.6 is transformed to 1.36. This establishes the equivalences under B.

For C, the translation $L \to A_1^{-1}LA_3$ takes 1.27 to 1.44.3 and 1.28 into 1.44.4. The transformation $L \to L^{-1}$ interchanges 1.44.2 and 1.44.1.2. The automorphism $C + (x, y) \to (x, x - y)$ interchanges 1.44.4 and 1.44.1.2. Finally the mapping $L \to L\ (12)(36)(48)(57)$ interchanges 1.44.3 and 1.44.4. This combination shows the equivalence of the planes in class C.

In class D, $L \to A_1^{-1}LA_3$ takes 1.26 to 1.44.8 and 1.30 to 1.44.7. The mapping $L \to (06)(17)(28)\ L(012)(375648)$ takes 4.5 into 4.3. We introduce new coordinates into the projective completion of 4.3 by making the replacements:

$$(2) \to (\infty)$$

$$(6) \to (0)$$

$$(0, 0) \to (0, 0)$$

$$(8, 4) \to (1, 1)$$

$$(7)\begin{pmatrix}0&1&2&3&4&5&6&7&8\\0&7&5&1&8&3&2&6&4\end{pmatrix} \to (1)\begin{pmatrix}0&7&5&4&2&6&8&3&1\\0&7&5&4&2&6&8&3&1\end{pmatrix}.$$

Under this change the plane 4.3 is found to be identical with plane 1.44.1.1. The transformation $L \to (354)(678)\ L(345)(687)$ takes 1.44.8 into 1.44.5. Also $L \to (345)(687)\ L(354)(678)$ takes 1.44.7 into 1.44.5. Finally, $L \to (13472685)\ L(1586-2743)$ takes 1.44.5 into 1.44.1.1. This completes the tables of equivalences.

Plane 1.44.1.1

*	0	1	2	3	4	5	6	7	8
*	1	2	0	4	5	3	7	8	6
*	2	0	1	5	3	4	8	6	7
*	3	4	5	6	7	8	0	1	2
*	4	5	3	7	8	6	1	2	0
*	5	3	4	8	6	7	2	0	1
*	6	7	8	0	1	2	3	4	5
*	7	8	6	1	2	0	4	5	3
*	8	6	7	2	0	1	5	3	4

*	0	2	1	6	8	7	3	5	4
*	1	0	2	7	6	8	4	3	5
*	2	1	0	8	7	6	5	4	3
*	3	5	4	0	2	1	6	8	7
*	4	3	5	1	0	2	7	6	8
*	5	4	3	2	1	0	8	7	6
*	6	8	7	3	5	4	0	2	1
*	7	6	8	4	3	5	1	0	2
*	8	7	6	5	4	3	2	1	0

Note: Starred lines are common to 1.44.1.1 and 1.44.1.2.

Plane 1.44.1.1—*continued*

*	0	3	6	2	7	4	1	8	5
	1	4	7	0	8	5	2	6	3
	2	5	8	1	6	3	0	7	4
*	3	6	0	7	4	2	8	5	1
	4	7	1	8	5	0	6	3	2
	5	8	2	6	3	1	7	4	0
*	6	0	3	4	2	7	5	1	8
	7	1	4	5	0	8	3	2	6
	8	2	5	3	1	6	4	0	7

*	0	4	8	5	2	6	7	3	1
	1	5	6	3	0	7	8	4	2
	2	3	7	4	1	8	6	5	0
	3	7	2	1	8	4	5	0	6
*	4	8	0	2	6	5	3	1	7
	5	6	1	0	7	3	4	2	8
	6	1	5	7	3	0	2	8	4
	7	2	3	8	4	1	0	6	5
*	8	0	4	6	5	2	1	7	3

*	0	5	7	8	3	2	4	1	6
	1	3	8	6	4	0	5	2	7
	2	4	6	7	5	1	3	0	8
	3	8	1	4	0	6	2	7	5
	4	6	2	5	1	7	0	8	3
*	5	7	0	3	2	8	1	6	4
	6	2	4	1	7	5	8	3	0
*	7	0	5	2	8	3	6	4	1
	8	1	3	0	6	4	7	5	2

*	0	6	3	1	5	8	2	4	7
	1	7	4	2	3	6	0	5	8
	2	8	5	0	4	7	1	3	6
*	3	0	6	8	1	5	7	2	4
	4	1	7	6	2	3	8	0	5
	5	2	8	7	0	4	6	1	3
*	6	3	0	5	8	1	4	7	2
	7	4	1	3	6	2	5	8	0
	8	5	2	4	7	0	3	6	1

*	0	7	5	4	6	1	8	2	3
	1	8	3	5	7	2	6	0	4
	2	6	4	3	8	0	7	1	5
	3	1	8	2	5	7	4	6	0
	4	2	6	0	3	8	5	7	1
*	5	0	7	1	4	6	3	8	2
	6	4	2	8	0	3	1	5	7
*	7	5	0	6	1	4	2	3	8
	8	3	1	7	2	5	0	4	6

*	0	8	4	7	1	3	5	6	2
	1	6	5	8	2	4	3	7	0
	2	7	3	6	0	5	4	8	1
	3	2	7	5	6	0	1	4	8
*	4	0	8	3	7	1	2	5	6
	5	1	6	4	8	2	0	3	7
	6	5	1	2	4	8	7	0	3
	7	3	2	0	5	6	8	1	4
*	8	4	0	1	3	7	6	2	5

Plane 1.44.1.2

*	0	1	2	3	4	5	6	7	8
*	1	2	0	4	5	3	7	8	6
*	2	0	1	5	3	4	8	6	7
*	3	4	5	6	7	8	0	1	2
*	4	5	3	7	8	6	1	2	0
*	5	3	4	8	6	8	2	0	1
*	6	7	8	0	1	2	3	4	5
*	7	8	6	1	2	0	4	5	3
*	8	6	7	2	0	1	5	3	4

*	0	2	1	6	8	7	3	5	4
*	1	0	2	7	6	8	4	3	5
*	2	1	0	8	7	6	5	4	3
*	3	5	4	0	2	1	6	8	7
*	4	3	5	1	0	2	7	6	8
*	5	4	3	2	1	0	8	7	6
*	6	8	7	3	5	4	0	2	1
*	7	6	8	4	3	5	1	0	2
*	8	7	6	5	4	3	2	1	0

*	0	3	6	2	7	4	1	8	5
	1	8	5	0	3	6	2	7	4
	2	7	4	1	8	5	0	3	6
*	3	6	0	7	4	2	8	5	1
	4	2	7	5	1	8	6	0	3

*	0	4	8	5	2	6	7	3	1
	1	7	3	8	0	4	6	5	2
	2	6	5	3	1	7	4	8	0
	3	1	7	4	8	0	2	6	5
*	4	8	0	2	6	5	3	1	7

Note: Starred lines are common to 1.44.1.1 and 1.44.1.2.

Plane 1.44.1.2—*continued*

	5	1	8	6	0	3	4	2	7
*	6	0	3	4	2	7	5	1	8
	7	4	2	8	5	1	3	6	0
	8	5	1	3	6	0	7	4	2
*	0	5	7	8	3	2	4	1	6
	1	6	4	5	7	0	3	2	8
	2	8	3	6	4	1	7	0	5
	3	2	8	1	6	4	5	7	0
	4	1	6	0	5	7	8	3	2
*	5	7	0	3	2	8	1	6	4
	6	4	1	7	0	5	2	8	3
*	7	0	5	2	8	3	6	4	1
	8	3	2	4	1	6	0	5	7
*	0	7	5	4	6	1	8	2	3
	1	4	6	3	8	2	5	0	7
	2	3	8	7	5	0	6	1	4
	3	8	2	5	0	7	1	4	6
	4	6	1	8	2	3	0	7	5
*	5	0	7	1	4	6	3	8	2
	6	1	4	2	3	8	7	5	0
*	7	5	0	6	1	4	2	3	8
	8	2	3	0	7	5	4	6	1

	5	2	6	7	3	1	0	4	8
	6	5	2	1	7	3	8	0	4
	7	3	1	0	4	8	5	2	6
*	8	0	4	6	5	2	1	7	3
*	0	6	3	1	5	8	2	4	7
	1	5	8	2	4	7	0	6	3
	2	4	7	0	6	3	1	5	8
*	3	0	6	8	1	5	7	2	4
	4	7	2	6	3	0	5	8	1
	5	8	1	4	7	2	6	3	0
*	6	3	0	5	8	1	4	7	2
	7	2	4	3	0	6	8	1	5
	8	1	5	7	2	4	3	0	6
*	0	8	4	7	1	3	5	6	2
	1	3	7	6	2	5	8	4	0
	2	5	6	4	0	8	3	7	1
	3	7	1	2	5	6	4	0	8
*	4	0	8	3	7	1	2	5	6
	5	6	2	0	8	4	7	1	3
	6	2	5	8	4	0	1	3	7
	7	1	3	5	6	2	0	8	4
*	8	4	0	1	3	7	6	2	5

Plane 1.44.6

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	5	3	4	8	6	7
3	4	5	6	7	8	0	1	2
4	5	3	7	8	6	1	2	0
5	3	4	8	6	7	2	0	1
6	7	8	0	1	2	3	4	5
7	8	6	1	2	0	4	5	3
8	6	7	2	0	1	5	3	4
0	3	6	2	7	4	1	8	5
1	8	3	5	6	2	0	7	4
2	6	4	0	8	5	7	1	3
3	0	8	7	4	1	2	5	6
4	2	5	8	1	0	6	3	7
5	1	7	6	2	3	8	4	0
6	4	0	1	3	7	5	2	8
7	5	2	4	0	8	3	6	1
8	7	1	3	5	6	4	0	2

0	2	1	6	8	7	3	5	4
1	0	2	8	7	6	5	4	3
2	1	0	7	6	8	4	3	5
3	8	7	0	5	4	6	2	1
4	7	6	5	0	3	2	1	8
5	6	8	4	3	0	1	7	2
6	5	4	3	2	1	0	8	7
7	4	3	2	1	5	8	0	6
8	3	5	1	4	2	7	6	0
0	4	8	5	2	6	7	3	1
1	7	4	2	3	8	6	5	0
2	8	5	3	0	7	1	4	6
3	1	6	4	8	2	5	0	7
4	0	7	1	6	5	3	8	2
5	2	3	0	7	1	4	6	8
6	3	2	7	5	0	8	1	4
7	6	1	8	4	3	0	2	5
8	5	0	6	1	4	2	7	3

Note· Starred lines are common to 1.44.1.1 and 1.44.1.2.

Plane 1.44.6—*continued*

0	5	7	8	3	2	4	1	6
1	6	5	7	2	4	3	0	8
2	7	3	6	4	0	5	8	1
3	2	4	1	0	6	8	7	5
4	1	8	2	5	7	0	6	3
5	0	6	3	1	8	7	2	4
6	8	1	4	7	5	2	3	0
7	3	0	5	8	1	6	4	2
8	4	2	0	6	3	1	5	7

0	6	3	1	5	8	2	4	7
1	5	6	0	4	7	8	3	2
2	3	7	4	1	6	0	5	8
3	7	0	8	2	5	1	6	4
4	8	2	6	3	1	7	0	5
5	4	1	7	0	2	6	8	3
6	0	5	2	8	3	4	7	1
7	2	8	3	6	4	5	1	0
8	1	4	5	7	0	3	2	6

0	7	5	4	6	1	8	2	3
1	4	7	3	8	0	2	6	5
2	5	8	1	7	3	6	0	4
3	6	2	5	1	7	4	8	0
4	3	1	0	2	8	5	7	6
5	8	0	2	4	6	3	1	7
6	1	3	8	0	4	7	5	2
7	0	4	6	5	2	1	3	8
8	2	6	7	3	5	0	4	1

0	8	4	7	1	3	5	6	2
1	3	8	6	0	5	4	2	7
2	4	6	8	5	1	3	7	0
3	5	1	2	6	0	7	4	8
4	6	0	3	7	2	8	5	1
5	7	2	1	8	4	0	3	6
6	2	7	5	4	8	1	0	3
7	1	5	0	3	6	2	8	4
8	0	3	4	2	7	6	1	5

Plane 1.4

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	5	3	4	8	6	7
3	4	5	6	7	8	0	1	2
4	5	3	7	8	6	1	2	0
5	3	4	8	6	7	2	0	1
6	7	8	0	1	2	3	4	5
7	8	6	1	2	0	4	5	3
8	6	7	2	0	1	5	3	4

0	2	1	6	8	7	3	5	4
1	0	2	8	7	6	5	4	3
2	1	0	7	6	8	4	3	5
3	5	7	0	2	4	6	8	1
4	3	6	5	0	2	7	1	8
5	4	8	2	3	0	1	7	6
6	8	4	3	5	1	0	2	7
7	6	3	4	1	5	8	0	2
8	7	5	1	4	3	2	6	0

0	3	5	2	1	6	4	8	7
1	6	4	5	2	8	3	7	0
2	8	3	0	4	7	5	1	6
3	0	8	1	6	5	7	2	4
4	7	1	8	5	0	6	3	2
5	1	7	4	8	2	0	6	3
6	4	2	7	0	3	8	5	1
7	5	0	6	3	1	2	4	8
8	2	6	3	7	4	1	0	5

0	4	3	8	2	1	7	6	5
1	8	5	7	3	2	6	0	4
2	7	4	6	0	5	1	8	3
3	1	6	2	5	7	8	4	0
4	0	7	3	1	8	2	5	6
5	6	1	0	7	3	4	2	8
6	3	0	1	8	4	5	7	2
7	2	8	5	4	6	0	3	1
8	5	2	4	6	0	3	1	7

0	5	4	1	7	2	8	3	6
1	7	3	2	6	4	0	5	8
2	6	5	3	8	0	7	4	1
3	8	1	4	0	6	2	7	5
4	1	8	6	2	3	5	0	7

0	6	8	7	5	4	2	1	3
1	3	7	6	4	0	8	2	5
2	5	6	8	1	3	0	7	4
3	7	2	5	8	1	4	0	6
4	8	0	2	7	5	3	6	1

Plane 1.4—*continued*

5	0	6	7	4	1	3	8	2
6	2	7	8	3	5	4	1	0
7	3	2	0	5	8	1	6	4
8	4	0	5	1	7	6	2	3

5	2	3	1	0	8	6	4	7
6	0	5	4	2	7	1	3	8
7	4	1	3	6	2	5	8	0
8	1	4	0	3	6	7	5	2

0	7	6	4	3	8	5	2	1
1	5	8	3	0	7	4	6	2
2	4	7	1	5	6	3	0	8
3	6	0	8	4	2	1	5	7
4	2	5	0	6	1	8	7	3
5	8	2	6	1	4	7	3	0
6	1	3	5	7	0	2	8	4
7	0	4	2	8	3	6	1	5
8	3	1	7	2	5	0	4	6

0	8	7	5	6	3	1	4	2
1	4	6	0	8	5	2	3	7
2	3	8	4	7	1	6	5	0
3	2	4	7	1	0	5	6	8
4	6	2	1	3	7	0	8	5
5	7	0	3	2	6	8	1	4
6	5	1	2	4	8	7	0	3
7	1	5	8	0	4	3	2	6
8	0	3	6	5	2	4	7	1

Plane 4.3

0	1	2	3	4	5	6	7	8
1	2	0	4	5	3	7	8	6
2	0	1	5	3	4	8	6	7
3	4	5	6	7	8	0	1	2
4	5	3	7	8	6	1	2	0
5	3	4	8	6	7	2	0	1
6	7	8	0	1	2	3	4	5
7	8	6	1	2	0	4	5	3
8	6	7	2	0	1	5	3	4

0	2	1	7	6	8	5	4	3
1	0	2	8	7	6	3	5	4
2	1	0	6	8	7	4	3	5
3	5	4	2	1	0	7	6	8
4	3	5	0	2	1	8	7	6
5	4	3	1	0	2	6	8	7
6	8	7	5	4	3	1	0	2
7	6	8	3	5	4	2	1	0
8	7	6	4	3	5	0	2	1

0	3	6	2	7	4	1	8	5
1	6	4	5	8	2	0	7	3
2	8	3	0	6	5	7	1	4
3	0	8	1	4	7	5	2	6
4	7	1	8	5	0	6	3	2
5	1	7	4	2	8	3	6	0
6	4	0	7	3	1	2	5	8
7	5	2	6	0	3	8	4	1
8	2	5	3	1	6	4	0	7

0	4	8	5	2	6	7	3	1
1	8	5	2	3	7	6	4	0
2	7	4	3	0	8	1	5	6
3	1	6	7	5	2	8	0	4
4	0	7	6	1	5	2	8	3
5	6	1	0	7	3	4	2	8
6	3	2	4	8	0	5	1	7
7	2	3	8	4	1	0	6	5
8	5	0	1	6	4	3	7	2

0	5	7	8	3	2	4	1	6
1	7	3	6	2	4	5	0	8
2	6	5	7	4	0	3	8	1
3	8	1	4	0	6	2	7	5
4	1	8	2	6	3	0	5	7
5	0	6	3	8	1	7	4	2
6	2	4	1	7	5	8	3	0
7	3	0	5	1	8	6	2	4
8	4	2	0	5	7	1	6	3

0	6	3	4	1	7	8	5	2
1	5	6	0	4	8	2	3	7
2	3	7	1	5	6	0	4	8
3	7	2	5	6	1	4	8	0
4	8	0	3	7	2	5	6	1
5	2	8	6	3	0	1	7	4
6	1	5	8	0	4	7	2	3
7	4	1	2	8	5	3	0	6
8	0	4	7	2	3	6	1	5

Plane 4.3—*continued*

0	7	5	1	8	3	2	6	4
1	4	7	3	6	0	8	2	5
2	5	8	4	7	1	6	0	3
3	6	0	8	2	5	1	4	7
4	2	6	5	0	7	3	1	8
5	8	2	7	1	4	0	3	6
6	0	3	2	5	8	4	7	1
7	1	4	0	3	6	5	8	2
8	3	1	6	4	2	7	5	0

0	8	4	6	5	1	3	2	7
1	3	8	7	0	5	4	6	2
2	4	6	8	1	3	5	7	0
3	2	7	0	8	4	6	5	1
4	6	2	1	3	8	7	0	5
5	7	0	2	4	6	8	1	3
6	5	1	3	2	7	0	8	4
7	0	5	4	6	2	1	3	8
8	1	3	5	7	0	2	4	6

7. Concluding remarks. The failure to find, in this extensive search, any planes beyond those known for many years strengthens the conjecture that the four planes are a complete set for order 9. This number is the smallest order still in doubt as to the list of possible planes and also is the smallest order with more than a single plane [2]. It is still not possible within the limits of present mechanical equipment to make a complete search for order 9 as was done for 8. For one thing, no list of latin squares of order 8 yet exists. It is planned to continue the search with other likely additive pencils beginning with the cyclic group.

The reader will have noted that, whereas the procedures used in the basic search were mechanized or easily mechanizable, the methods of section 6 were more involved and clearly *ad hoc*. No satisfactory mechanical way to identify two isomorphic planes exists whether they be presented by a coordinate system or by an incidence matrix. The preparation of such a method is an interesting question.

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Functional Approximations and Dynamic Programming

By Richard Bellman and Stuart Dreyfus

1. Introduction. In this note we wish to indicate some ways in which the theory of approximation can be used to increase the range of present day computers. Although we are primarily interested in applying these techniques to the functional equations occurring in the theory of dynamic programming, [1], it should be noted that these same methods are applicable, and even more readily, to the classical functional equations of mathematical physics.

What we wish to do is to trade additional computing time, which is expensive, for additional memory capacity, which does not exist.

2. Dimensionality Difficulties. A typical problem arising in the theory of control processes is that of maximizing a functional of the form

$$J(y) = \int_0^T g(x, y)\, dt, \tag{1}$$

where x and y are N-dimensional vectors related by the differential equation

$$\frac{dx}{dt} = h(x, y), \qquad x(0) = c. \tag{2}$$

As we have discussed elsewhere at some length, [2], [3], questions of this nature, although nominally within the domain of the calculus of variations, in actuality cannot be reduced to the point of numerical solution by means of classical techniques.

Writing

$$\operatorname*{Max}_{y} J(y) = f(c, T), \tag{3}$$

the theory of dynamic programming replaces the foregoing variational problem by that of solving the nonlinear partial differential equation

$$\frac{\partial f}{\partial T} = \operatorname*{Max}_{v} [g(c, v) + f(h(c, v), \partial f/\partial c)], \tag{4}$$

$$f(c, 0) = 0,$$

where

$$\frac{\partial f}{\partial c} = \left(\frac{\partial f}{\partial c_1}, \frac{\partial f}{\partial c_2}, \cdots, \frac{\partial f}{\partial c_N}\right). \tag{5}$$

For computational purposes, it is often convenient to use the approximate difference equation

$$f(c, T + \Delta) = \operatorname*{Max}_{v} [g(c, v)\Delta + f(c + h(c, v)\Delta, T)]. \tag{6}$$

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In terms of the capacities of modern computers, we have an extremely efficient algorithm if $N = 1$, a scalar problem, and a feasible algorithm if $N = 2$. If $N = 3$ or more, we face fast memory difficulties if we attempt to proceed in a routine fashion.

The reason for this is the following. To store a function of N variables in the usual way, we tabulate the values of the function at a set of lattice points within the domain of interest. If there are M different possible values of c_1, of c_2, and so on, the total number of grid points will be M^N. For $M = 100$, and $N = 3$, this yields a quantity outside of present capabilities.

In various papers, [4], [5], we have indicated some methods which enable us to circumvent these difficulties. These methods combine analytic devices with the method of successive approximation. In this paper, we wish to present a new method, based upon approximation techniques, which appears to have wide applicability.

3. One-dimensional Case. In order to illustrate the application of the method in its simplest form, let us consider the problem of determining the sequence of functions $\{f_n(c)\}$, $n = 1, 2, \cdots$, given by the recurrence relation

$$
\begin{aligned}
f_1(c) &= \operatorname*{Max}_v g(c, v), \\
f_{n+1}(c) &= \operatorname*{Max}_v [g(c, v) + f_n(h(c, v))].
\end{aligned} \tag{1}
$$

Let us suppose that c takes values only over $[-1, 1]$ and that the function $h(c, v)$ similarly assumes values over this interval for all c-values and all permissible v-values.

The standard approach involves a grid of values in the interval $[-1, 1]$ where the number of grid-points depends upon the accuracy that we desire. Let us proceed in a different manner. In place of considering that the function is determined by the set of grid-points, we shall consider the function to be determined by a Fourier expansion in terms of a suitable orthonormal set. For the interval $[-1, 1]$, a convenient set is the set of normalized Legendre polynomials. Thus, for some fixed value R, we write

$$
f_n(c) = \sum_{k=0}^{R} a_{k,n} P_k(c). \tag{2}
$$

This is, of course, an approximation comparable with that of using a finite number of grid-points.

We could envisage using a power series expansion rather than an expansion in terms of orthogonal functions. Generally speaking, an orthogonal expansion is to be preferred, both on the grounds of accuracy and the grounds of ease of determination of the coefficients. As far as the calculation of the values of $P_k(c)$ are concerned, the simple three-term recurrence relations connecting the successive members of the sequence of Legendre polynomials make the computation of these values not much more difficult than that of the calculation of the powers c^k.

The function $f_n(c)$ is now replaced, as far as storage in the computer is concerned, by the set of coefficients

$$
A_n = [a_{0,n}, a_{1,n}, \cdots, a_{R,n}]. \tag{3}
$$

Whenever we wish a value of $f_n(c)$, we compute it by means of the approximate relation in (2). Consequently, in computing the values of $f_{n+1}(c)$ by way of (1), we require only the storage of the set of values given by A_n.

There is, however, a difficulty in this approach. The coefficients are determined by the relations

$$a_{k,n} = \int_{-1}^{1} f_n(c)P_k(c)\,dc. \tag{4}$$

How are we going to calculate them? If we replace the integral by a simple Riemann sum, we are led back to the necessity for tabulating the values of $f_n(c)$ at a set of grid-points.

It is here that we invoke the theory of mechanical quadrature. In place of evaluating the integrals in (4) by means of Riemann sums, we use an interpolation formula of the following form:

$$\int_{-1}^{1} f_n(c)P_k(c)\,dc = \sum_{j=1}^{S} a_j f_n(c_j)P_k(c_j), \tag{5}$$

where the a_j and c_j are carefully chosen.

If the quantities c_j are chosen to be the zeros of the Legendre polynomial of degree S, and the coefficients a_j are chosen to be the Christoffel numbers, the formula in (5) is exact if the integrand $f_n(c)P_k(c)$ is a polynomial of degree $2S - 1$.

Since the quantities a_j and c_j are tabulated, and in any case, could easily be determined in advance, the amount of effort entailed depends upon the choice of S. Since our original approximation, contained in (2), is equivalent to that of assuming that $f_n(c)$ is a polynomial of degree R, it would be reasonable to take S equal to R. However, there is no necessity for doing this.

If we use the equation in (5) to compute the values of the coefficient set A_n, it is clear that at each stage we require only the values $f_n(c_j)$, $j = 1, 2, \cdots, R$.

The computation then proceeds in the following fashion. Given the values of $f_n(c_j)$, we compute the coefficient set A_n. Using these coefficients we can determine the values of $f_n(h(c, v))$ which occur in (1) in the course of computing the new values $f_{n+1}(c_j)$.

4. Discussion. Apart from the fixed set of instructions, and the values such as $a_jP_k(c_j)$ which are determined at the beginning of the process, we require the set $A_n = [a_{0,n}, a_{1,n}, \cdots, a_{R,n}]$ to be retained in the fast memory at the nth stage. This is a set of $(R + 1)$ values.

On the other hand, a grid size of δ over $[-1, 1]$ would require $1/\delta$ values. In one dimension, the difference between $R + 1$ and $1/\delta$ is not particularly important, and the great amount of additional computation required by the method described above can more than outweigh this advantage.

Consider, however, the two-dimensional case. The straight-forward approach based upon a grid size of δ in the c_1 and c_2 intervals requires $(1/\delta)^2$ values of $f_n(c_1, c_2)$. On the other hand, if we set

$$f_n(c_1, c_2) = \sum_{i,j=0}^{R} a_{ij,n}P_i(c_1)P_j(c_2), \tag{1}$$

and proceed as above, we require only $R(R + 1)/2$ values, the coefficients $a_{ij,n}$, $i, j = 0, 1, \cdots, R$.

Proceeding to three dimensions, we compare $(1/\delta)^3$ and $R(R+1)(R+2)/6$, the number of coefficients $a_{ijk,n}$. Let us use some typical values of δ and R and compare the values, say $\delta = .01$ and $R = 5, 10$.

Dimension	$(1/\delta)^N$	$R = 5$	$R = 10$
1	100	6	11
2	10^4	21	55
3	10^6	56	286
4	10^8	126	1001
5	10^{10}	252	3003

We see that variational problems involving four and five state variables which are completely untouchable by direct methods are within the scope of the method we have outlined. Combining this method with the Lagrange multiplier technique, and the method of successive approximations, we have a way of attacking previously impregnable problems.

5. A Numerical Example. Consider the problem of determining the sequence $\{v_n\}$ so as to minimize the function

$$F_N = \sum_{k=1}^{N} u_k^2 + \lambda \sum_{k=1}^{N} v_k^2, \tag{1}$$

where

$$\begin{aligned} &\text{(a)}\ u_{n+1} = 2u_n - u_n^2 + v_n,\ u_0 = c, \\ &\text{(b)}\ v_n \text{ must be chosen so as to keep } u_{n+1} \text{ within the interval } [-1, 1]. \end{aligned} \tag{2}$$

Introducing for $N \geqq 1$ and $-1 \leqq c \leqq 1$, the function

$$f_N(c) = \operatorname*{Min}_{\{v\}} F_N, \tag{3}$$

we derive the recurrence relation

$$\begin{aligned} f_N(c) &= \operatorname*{Min}_{v} [c^2 + \lambda v^2 + f_{N-1}(2c - c^2 + v)], \qquad N \geqq 2, \\ f_1(c) &= c^2. \end{aligned} \tag{4}$$

This yields a simple computational determination of the sequence $\{f_N(c)\}$.

As a test of the foregoing method, this sequence was first determined by the usual method based upon a grid of values over $[-1, 1]$, and then following the procedure described above, using the approximation

$$f_N(c) = \sum_{k=0}^{6} a_{k,N} P_k(c), \tag{5}$$

and a value of $S = 14$ in (3.5).

Let $\{f_k(c)\}$ denote the sequence determined using the grid, and $\{f_k^*(c)\}$ the sequence obtained via Legendre polynomials. A comparison of values is given below for $k = 6$.

c	$f_6^*(c)$	$f_6(c)$
1.0	1.782	1.77
.8	1.370	1.36
.2	.153	.145
0.0	.006	0.0
−.2	.202	.20
−.8	4.876	4.89
−1.0	8.666	8.67

As can be seen, the agreement is quite good.

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The Influence of Critical Boundary Conditions on Finite Difference Solutions of Two Point Boundary Value Problems

By A. R. Mitchell

1. Introduction. It is often assumed in the solution of two point boundary problems by finite difference methods that an accurate answer is obtained provided the mesh length is small and the numerical calculations are carried out with sufficient accuracy. The object of the present note is to illustrate by means of worked out examples the extent of the disagreement which often exists, even for small mesh lengths, between corresponding exact solutions of ordinary differential and difference equations. Although the differential equation chosen for purposes of illustration in the present paper is a particularly simple one, the difficulties outlined are quite general and arise in the numerical solution of more complicated equations subject to two point boundary conditions.

The equation considered is

$$\frac{d^2y}{dx^2} + p\sqrt{k}\,\frac{dy}{dx} + ky = 0, \qquad (k > 0) \tag{1}$$

with the associated second order central difference replacement

$$(1 + \tfrac{1}{2}ph)y_{r+1} - (2 - h^2)y_r + (1 - \tfrac{1}{2}ph)y_{r-1} = 0, \tag{2}$$

where p, k are constants, y_{r+1}, y_r, y_{r-1} are the values of y at $x = (r + 1)H$, rH, $(r - 1)H$ respectively, and $h = \sqrt{k}H$ (where H is the mesh length). In all cases the range of the problem is from $x = 0$ to $x = L$ where

$$L = (N + 1)H, \tag{3}$$

and N is the number of internal nodes. Corresponding exact solutions of (1) and (2) are compared for two types of boundary condition.

2. Function given at both ends of the range. Consider the boundary conditions $y = 0$ at $x = 0$ and $y = Y$ at $x = L$. The solution of (1) becomes

$$y = \frac{\sin \tfrac{1}{2}\sqrt{4 - p^2}\sqrt{k}x}{\sin \tfrac{1}{2}\sqrt{4 - p^2}\,l}\, e^{\frac{1}{2}p(l-\sqrt{k}x)}Y \tag{4a}$$

if $p^2 < 4$, and

$$y = \frac{\sinh \tfrac{1}{2}\sqrt{p^2 - 4}\sqrt{k}x}{\sinh \tfrac{1}{2}\sqrt{p^2 - 4}\,l}\, e^{\frac{1}{2}p(l-\sqrt{k}x)}Y \tag{4b}$$

if $p^2 > 4$, where $l = \sqrt{k}\,L$. No finite solution exists for general x when

$$l = \frac{2r\pi}{\sqrt{4 - p^2}}. \qquad (r = 1, 2, 3, \cdots) \tag{5}$$

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The solution of (2) becomes

$$y_r = \left(\frac{2 + ph}{2 - ph}\right)^{\frac{N+1-r}{2}} \frac{\sin r\theta}{\sin (N+1)\theta} Y \tag{6a}$$

where

$$\cos \theta = \frac{2 - h^2}{\sqrt{4 - p^2h^2}}$$

if $p^2 + h^2 < 4$, and

$$y_r = \frac{\lambda^r - \mu^r}{\lambda^{N+1} - \mu^{N+1}} Y \qquad (r = 1, 2, \cdots N) \tag{6b}$$

where

$$\lambda, \mu = \frac{2 - h^2 \pm h\sqrt{p^2 + h^2 - 4}}{2 + ph}$$

if $p^2 + h^2 > 4$. No finite solution exists at general nodes for ranges satisfying

$$l = \sqrt{2}(N+1)\left[1 - \sqrt{1 - \frac{p^2l^2}{4(N+1)^2}} \cos \frac{K\pi}{N+1}\right]^{\frac{1}{2}}. \qquad (K = 1, 2, \cdots N) \tag{7}$$

Ranges for which either the differential or the difference equation has no finite solution are said to be critical.

It can be shown that for a prescribed value of K, a critical range of the difference equation given by (7) either approaches a critical range of the differential equation given by (5) or tends to infinity as N tends to infinity. Due to the presence of the mesh length, however small, in a difference calculation, there is never exact correspondence between critical ranges of differential and difference equations, and an attempt is now made to determine the effect of this lack of correspondence in critical range on the agreement between exact solutions of differential and associated difference equations.

Calculations are carried out for $p = 0$, $\sqrt{2}$, 1.9, 2.0, $\sqrt{4.99}$ and for $h = 0.1$, 0.2. Values of y are obtained from (4) at $x = L/2$, and from (6) at the node $r = (N+1)/2$. The difference between corresponding exact solutions of the differential and difference equations is expressed as a percentage of the solution of the differential equation, and the resulting values are quoted in Table 1. It should be pointed out that when $x = L/2$, (5) requires modification, and in fact no finite solution of the differential equation exists for

$$l = \frac{2r\pi}{\sqrt{4 - p^2}}. \qquad (r = 1, 3, 5, \cdots)$$

Also the sign in front of the percentage differences is plus or minus depending on whether the difference equation solution is greater or less than the differential equation solution.

There are the following points to be noted in the table:

(1) The values of l for which the differential equation has no finite solution are indicated, and it is clear that poor agreement exists between the exact solutions of the differential and difference equations in the vicinity of these critical values.

TABLE 1. *Values of* $\dfrac{100(y_{(N+1/2)} - y_{L/2})}{y_{L/2}}$

$p = 0$	l	2	3	3.1	*3.14*	3.2	4	6	8	9	9.4	*9.43*	9.5	10
	$h = 0.1$	+.06	+.89	+3.20		+2.23	+.18	+.02	−.20	−.87	−18.72		−4.98	−.70
	$h = 0.2$	+.26	+3.67			+8.39	+.72	+.08	−.78	−3.62	−173.31			−2.75
$p = \sqrt{2}$	l	2	4	4.4	*4.44*	4.6	5	8	12	13.2	*13.33*	13.6	14	16
	$h = 0.1$	+.17	+.99	+9.81		+2.18	+.43	−.40	−1.41	−10.12		−3.20	−.84	+.61
	$h = 0.2$	+.69	+4.11	+54.82		+8.03	+1.72	−1.62	−5.89	−57.55		−11.55	−3.25	+2.50
$p = 1.9$	l	2	4	6	8	9.2	10	*10.06*	10.8	12	14	16		
	P no.	168	768	2,244	6,885	18,090	189,523		33,730	14,318	8,710	6,910		
	$h = 0.1$	+.21	+.22	−.13	−1.72	−6.47	−55.59		−15.02	−6.78	−4.58	−3.86		
	$h = 0.2$	+.82	+.83	−.71	−7.17	−22.63	−84.03		−114.05	−35.66	−21.77	−17.85		
$p = 2.0$	l	2	4	6	8	10	12	14	16					
	P no.	161	646	1,446	2,555	3,978	5,671	7,615	9,870					
	$h = 0.1$	+.21	+.17	−.12	−.66	−1.44	−2.44	−3.65	−5.06					
	$h = 0.2$	+.85	+.67	−.52	−2.61	−5.50	−9.05	−13.10	−17.51					
$p = \sqrt{4.99}$	l	2	4	6	8	10	12	14	16					
	P no.	147	460	759	983	1,140	1,247	1,318	1,375					
	$h = 0.1$	+.22	+.11	−.09	−.23	−.32	−.43	−.53	−.61					

(2) As p approaches the value 2 from below, agreement between the exact solutions deteriorates and when $p = 1.9$, for example, the error is in excess of 2% when $h = 0.1$ for any $l > 8.2$.

(3) Once p has attained the value 2, the critical ranges of the differential equation are at infinity and the percentage errors increase monotonically with l.

(4) The P-condition numbers are given for $p = 1.9, 2.0, \sqrt{4.99}$ when $h = 0.1$. The magnitude of the P number indicates the order of difficulty of obtaining an accurate numerical solution of the appropriate difference equation. The numbers are large in the vicinity of critical ranges.

3. Function and derivative given, one at either end of the range. Next consider the boundary conditions $y = 0$ at $x = 0$ and $y' + \alpha y = \beta$ at $x = L$. The solution of (1) becomes at $x = L$,

$$y\sqrt{k} = \frac{2\beta}{\delta - p + \sqrt{4 - p^2}\cot\sqrt{4 - p^2}\,\tfrac{1}{2}l} \tag{8a}$$

if $p^2 < 4$, and

$$y\sqrt{k} = \frac{2\beta}{\delta - p + \sqrt{p^2 - 4}\coth\sqrt{p^2 - 4}\,\tfrac{1}{2}l} \tag{8b}$$

if $p^2 > 4$, where $l = \sqrt{k}\,L$ and $\delta = 2\alpha/\sqrt{k}$. The solution ceases to be finite when

$$l = \frac{2(r\pi - \phi)}{\sqrt{4 - p^2}} \qquad (r = 1, 2, \cdots) \tag{9a}$$

where

$$\phi = \tan^{-1}\frac{\sqrt{4 - p^2}}{\delta - p}$$

if $p^2 < 4$, and when

$$l = \frac{1}{\sqrt{p^2 - 4}}\log_e\frac{\delta - p - \sqrt{p^2 - 4}}{\delta - p + \sqrt{p^2 + 4}} \tag{9b}$$

if $p^2 > 4$. It should be noted that the derivative condition at $x = L$ causes critical ranges to exist for $p^2 > 4$ as well as for $p^2 < 4$.

If the derivative condition at $x = L$ is replaced by

$$2y_N - [(2 - h^2) + (2 + ph)\tfrac{1}{2}\delta h]\,y_{N+1} + (2 + ph)h\frac{\beta}{\sqrt{k}} = 0,$$

the solution of (2) becomes at the node $r = N + 1$,

$$y\sqrt{k} = -\frac{\beta h(2 + ph)}{[h^2 - (2 + ph)\tfrac{1}{2}\delta h - 2] + 2\left(\dfrac{2 + ph}{2 - ph}\right)^{\frac{1}{2}}\dfrac{\sin N\theta}{\sin(N + 1)\theta}} \tag{10a}$$

where

$$\cos 2\theta = \frac{1 + (\frac{1}{4}p^2 - 2)h^2 + \frac{1}{2}h^4}{1 - \frac{1}{4}p^2h^2}$$

if $p^2 + h^2 < 4$, and

$$y\sqrt{k} = -\frac{\beta h(2 + ph)}{[h^2 - (2 + ph)\frac{1}{2}\delta h - 2] + 2\dfrac{\lambda^N - \mu^N}{\lambda^{N+1} - \mu^{N+1}}} \tag{10b}$$

where

$$\lambda, \mu = \frac{2 - h^2 \pm h\sqrt{p^2 + h^2 - 4}}{2 + ph}$$

if $p^2 + h^2 > 4$. No finite solution exists when

$$l = 2(N + 1)\left[1 - \sqrt{1 - \frac{p^2l^2}{4(N + 1)^2}}\cos\frac{K\pi - \phi}{N + 1}\right]^{\frac{1}{2}} \quad (K = 1, 2, \cdots N) \tag{11a}$$

where

$$\tan\phi = \frac{[(4 - h^2) - p^2]^{\frac{1}{2}}}{\delta(1 - \frac{1}{4}p^2h^2) + \frac{1}{2}p(h^2 - 2)}$$

if $p^2 + h^2 < 4$, and when

$$\left[\frac{h^2 - h\sqrt{p^2 + h^2 - 4} - 2}{h^2 + h\sqrt{p^2 + h^2 - 4} - 2}\right]^{N+1} = \frac{\delta(1 - \frac{1}{4}p^2h^2)h + \frac{1}{2}p(h^2 - 2)h - \sqrt{p^2 + h^2 - 4}}{\delta(1 - \frac{1}{4}p^2h^2)h + \frac{1}{2}p(h^2 - 2)h + \sqrt{p^2 + h^2 - 4}} \tag{11b}$$

if $p^2 + h^2 > 4$. Again, it can be shown that the critical ranges of the difference equation given by (11) approach the corresponding critical ranges of the differential equation given by (9) or tend to infinity as N tends to infinity.

Calculations are carried out for $p = 1.9, 2, 3$ with $h = 0.1$. A range of values of δ is covered for each value of p. The values of $y\sqrt{k}/\beta$ for the differential and difference equations are obtained from (8a) and (10a) respectively when $p = 1.9$, and from (8b) and (10b) respectively when $p = 2, 3$. In Table 2, the percentage differences between corresponding exact solutions of the differential and difference equations are quoted for $p = 2, 3$, whereas the actual differences between corresponding exact solutions are quoted in units of .0001 for $p = 1.9$. This is because the exact solution of the differential equation for $p = 1.9$ is sometimes so small that the percentage differences are misleadingly high.

There are several points arising from Table 2.

(1) An entry ∞ in the table means that the differential equation has no finite solution for the appropriate values of l and δ. As in Table 1, poor agreement exists between exact solutions in the vicinity of these critical values.

(2) When $p = 1.9$, critical values occur periodically for each value of δ, but when $p = 2, 3$ there is at most one critical range for each value of δ. For example

TABLE 2. *Values of* $(y_{N+1} - y_L)$

	l	.6	1.0	1.5	2.0	4.0	6.0	8.0	8.2	8.5	8.8	9.1	9.4	9.7	10	11	12	14
$p = 1.9$	$\delta = 0$	−76	$-\infty$	+29	+31	+10	−17	−82	−95	−122	−158	−211	−293	−432	−692	$-\infty$	−1082	−122
	$\delta = 1$	−25	−35	+114	$+\infty$	+57	−60	−205	−229	−270	−321	−386	−473	−591	−761		$-\infty$	−762
	$\delta = 3$	−7	−4	+2	+9	+16	−87	−6,607	−27,426	$-\infty$	−14,119	−4,451	−2,229	−1,365	−937		−248	−193
	$\delta = 6$	−3	−1	0	0	+2	−5	−56	−77	−131	−256	−668	−4,718	$-\infty$	−1,353		−40	−18

Values of $\dfrac{100(y_{N+1} - y_L)}{y_L}$

	l	.5	.9	1.0	1.1	1.5	1.9	2.0	2.1	3.0	10	20	∞
$p = 2$	$\delta = 0$	−.54	−1.17	$-\infty$	−.51	+.13	+3.10	+.17	+3.54	+.50	−.42	−1.19	−4.76
	$\delta = 1$	−.36		−.17		+.25		$+\infty$			−.95	−2.52	−10.00
	$\delta = 2$	−.28		−.08		+.07		+.17			−3.64	−18.14	$-\infty$
	$\delta = 3$	−.22		−.05		+.01		+.08			−.63	−1.97	−8.34

	l	.2	.6	.8	.96	.9	1.0	1.5	1.9	2.0	2.5	∞
$p = 3$	$\delta = 0$	−1.22	−.98	−1.14	$-\infty$	+.07	+.45	+.31	$+\infty$	+.18	+.10	0
	$\delta = .70$	−1.14	−.53				+.22	+1.97		+8.37	+1.08	−.07
	$\delta = .764$	−1.11	−.51				+.19	+1.18		+2.22	+3.31	$+\infty$
	$\delta = 1$	−1.06	−.44				+.13	+.48		+.39	+.20	−.02

when $p = 2$, as δ increases from 0 to 2, the value of the critical range increases from 1 to ∞. For any value of $\delta > 2$, there is no critical range.

(3) The lack of agreement between exact solutions in the vicinity of critical ranges seems to be less pronounced than was the case in Table 1.

4. The effect of reducing the mesh length on the accuracy of finite difference solutions. So far, exact solutions of the difference equation have been obtained for mesh lengths of 0.1 or 0.2. Calculations are now carried out to show the effect of reducing the mesh length on the accuracy of finite difference solutions.

Considered the differential equation

$$\frac{d^2y}{dx^2} + p\sqrt{k}\frac{dy}{dx} - ky = 0 \qquad (k > 0)$$

subject to the boundary conditions $y = 0$ at $x = 0$ and $y' + \alpha y = \beta$ at $x = L$. The values of $y\sqrt{k}/\beta$ are calculated from the conventional second order central difference replacement for a range of values of δ and N when $\sqrt{k}L = 4$ and $p = -6$, and are shown in Table 3.

The differential equation itself has no finite solution when $\delta = -12.32$, whilst the difference equation has no finite solution at the following values of δ and N

N	3	10	12	15	31	63	127	∞
δ	+1.95	+66.98	−85.28	−28.61	−14.40	−12.79	−12.44	−12.32.

It is the lack of correspondence between critical conditions of the differential and difference equations which causes poor agreement to exist between the exact solutions. For example, when $\sqrt{k}L = 4$, $\delta = -15$, the difference solution at $x = L$ is $-.1180$ for $N = 3$, and as N tends to 29, the solution increases monotonically to $+\infty$. For $N > 29$, the solution increases from $-\infty$ reaching the value $-.9034$ when $N = 63$. Thus although the mesh length $h(=\sqrt{k}H)$ is as small as $\frac{1}{16}$, the exact solution of the difference equation is approximately 20% less than the exact solution of the differential equation. Of course, for values of δ less than -15, the infinite step in the solution of the difference equation occurs at a value of N less than 29, (see Table 3) and so the solution of the difference equation for $N = 63$ is correspondingly more accurate. When $\delta = -30$, for example, the error is reduced to 2.65%, which nevertheless is still appreciably large.

TABLE 3. *Values of* $\frac{y\sqrt{k}}{\beta}$

N	δ								
	−9	−12	−12.32	−15	−18	−21	−24	−27	−30
3	−.1826	−.1433		−.1180	−.1002	−.0871	−.0771	−.0691	−.0626
10	−.0263	−.0253		−.0244	−.0235	−.0227	−.0220	−.0213	−.0206
12	+.0262	+.0273		+.0284	+.0297	+.0311	+.0326	+.0343	+.0362
									±∞ (14)
15	+.1020	+.1204		+.1469	+.1885	+.2628	+.4338	+1.2420	−1.4391
				±∞ (29)	±∞ (23)	±∞ (20)	±∞ (18)	±∞ (16)	
31	+.3706	+.8343		−3.3183	−.5551	−.3029	−.2083	−.1587	−.1282
63	+.5282	+2.5442		−.9034	−.3836	−.2435	−.1784	−.1407	−.1162
∞	+.6016	+6.1623	±∞	−.7476	−.3524	−.2305	−.1713	−.1363	−.1132

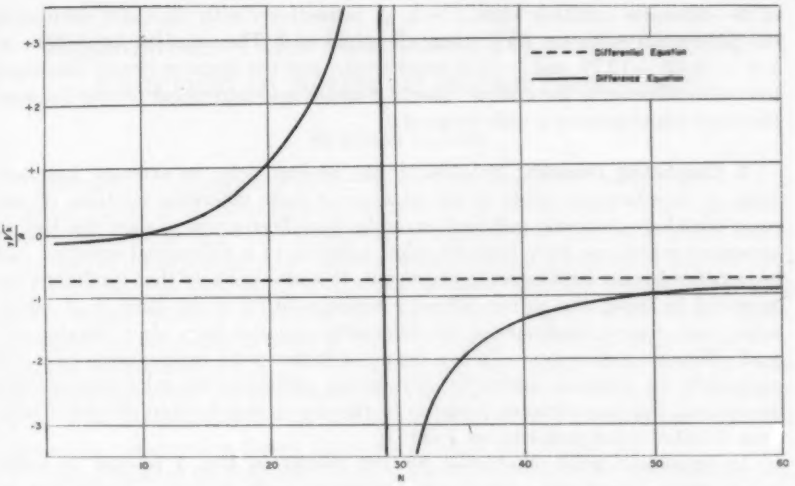

FIG. 1.—Convergence of finite difference solution toward solution of differential equation.

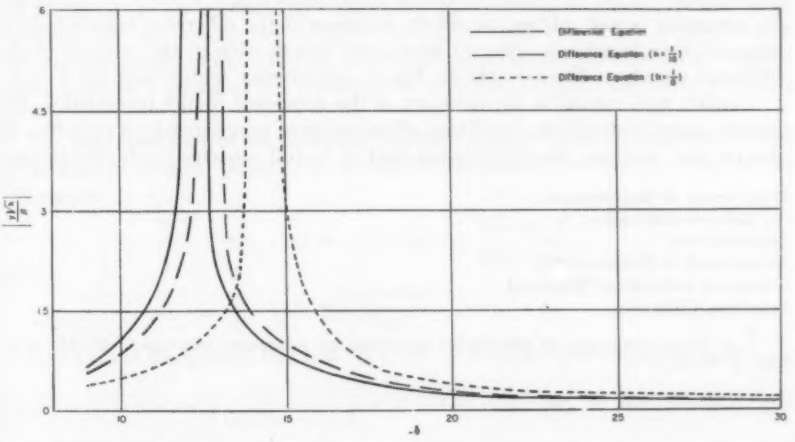

FIG. 2.—Comparison of finite difference solution with solution of differential equation for varying δ.

Two figures are now given to illustrate some of the results contained in Table 3. Fig. 1 illustrates the manner in which the exact solution of the difference equation converges towards the exact solution of the differential equation as the number of internal nodes is increased. The outstanding feature is the infinite step in the solution. Reduction of the mesh length only improves the solution of the difference equation once the infinite step has been passed. Fig. 2 compares the exact solutions

of the difference equation when $h = \frac{1}{8}, \frac{1}{16}$ respectively with the exact solution of the differential equation, for a range of values of δ. The solutions are infinite at $\delta = -14.40, -12.79$, and -12.32 respectively, and the diagram clearly illustrates how this difference in the critical value of δ causes poor agreement to exist between the exact solutions over a wide range of δ.

5. Concluding remarks. Although in the present note, no attempt has been made at an exhaustive study of the accuracy of finite difference solutions of two point boundary problems, sufficient examples have been given to show the lack of agreement which can exist between exact solutions of a differential equation and its associated central difference replacement. It is also unlikely that conditions are improved by using alternative difference schemes either of the same or of higher order, since critical conditions of the differential equation have their counterparts in all difference replacements. In fact the use of higher order replacements may well complicate the situation further by introducing additional critical conditions into the system. This was certainly found to be the case in step-by-step difference solutions of initial value problems by Todd [1].

An important point illustrated by the results in Fig. 1 is that in order to guarantee a reasonable answer using difference methods it is necessary but by no means sufficient for the exact solution of the difference equation to tend to the exact solution of the differential equation as the mesh length tends to zero. In fact, the computor is well advised to obtain solutions of the difference equation for a range of mesh lengths in order to detect any infinite step in the solution of the difference equation. For example, in Fig. 1, convergence occurs only for $h < \frac{2}{15}$.

Finally, with regard to the accuracy of the numerical results presented in the present paper, it should be stated that all calculations were carried out on a 10 x 10 electric desk machine, the latter being used to its full capacity in all calculations.

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Expansion of the Confluent Hypergeometric Function in Series of Bessel Functions

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Abstract. An expansion of the confluent hypergeometric function $\Phi(a, c, z)$ in series of functions of the same kind has been given by Buchholz [1]. By specialization of some quantities, there is obtained an expansion in series of modified Bessel functions of the first kind, $I_\nu(z)$, where ν depends on the parameter a. Tricomi [2, 3] has developed two expansions of similar type where both the order and argument of the Bessel functions depend on the parameters a and c. In the present paper, we derive an expansion in series of Bessel functions of integral order whose argument is independent of a and c. Our expansion is advantageous for many purposes of computation since the parameters and variable of $\Phi(a, c, z)$ appear in separated form. Also, for desk calculation, extensive tables of $I_n(z)$ are available, while for automatic computation Bessel functions are easy to generate [4].

Special cases of the confluent function, such as the incomplete gamma function, are also studied. For the class of transcendents known as the error functions, including the Fresnel integrals, it is shown that our expansion coincides with that of Buchholz [1]. By specializing a parameter and passing to a limit, we derive expansions for the exponential integral and related functions. Other expansions for the error and exponential integrals are derived on altogether different bases. Finally, some numerical examples are presented to manifest the efficiency of our formulas.

1. Representation of the Confluent Hypergeometric Function in Series of Bessel Functions of Integral Order. The confluent hypergeometric function is defined by the series

$$\Phi(a, c, z) \equiv {}_1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(c)_k k!}, \tag{1.1}$$

$$(a)_k = \frac{\Gamma(a + k)}{\Gamma(a)},$$

where $\Gamma(z)$ is the gamma function. We also have the integral representation

$$\Phi(a, c, z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 e^{zt} t^{a-1} (1 - t)^{c-a-1}\, dt, \qquad R(c) > R(a) > 0, \tag{1.2}$$

and Kummer's formula

$$\Phi(a, c, z) = e^z \Phi(c - a, c, -z). \tag{1.3}$$

For these and other properties of the confluent function, see [5].

Let $t = \sin^2\theta$ in the integrand of (1.2). Using the Jacobi expansion [6]

$$e^{(z/2)\cos 2\theta} = \sum_{k=0}^{\infty} \epsilon_k \cos 2k\theta I_k(z/2), \quad \epsilon_0 = 1, \quad \epsilon_k = 2, \quad k > 0, \tag{1.4}$$

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(1.2) becomes

$$\Phi(a,c,z) = \frac{2\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\, e^{z/2} \sum_{k=0}^{\infty} (-1)^k \epsilon_k I_k(z/2)$$

$$\cdot \int_0^{\frac{\pi}{2}} \cos 2k\theta (\sin\theta)^{2a-1} (\cos\theta)^{2c-2a-1}\, d\theta. \tag{1.5}$$

Since

$$\cos 2k\theta = (-1)^k \sum_{m=0}^{k} \frac{(-1)^m \binom{k}{m} (k)_m (\cos\theta)^{2m}}{(1/2)_m}, \tag{1.6}$$

where $\binom{k}{m}$ is the binomial coefficient, and

$$\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1}\, d\theta = \frac{1}{2}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad R(x) > 0, \qquad R(y) > 0, \tag{1.7}$$

we find that

$$\Phi(a, c, z) = e^{z/2} \sum_{k=0}^{\infty} (-1)^k \epsilon_k R_k(a, c) I_k(z/2), \tag{1.8}$$

where $R_k(a, c)$ is conveniently expressed in hypergeometric form [7] as

$$R_k(a, c) = {}_3F_2(-k, k, a; c, \tfrac{1}{2}; 1). \tag{1.9}$$

An alternative expression for $R_k(a, c)$ is readily reduced from a result in [8]. We have

$$\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} R_k(a,c) = G_k(a,c) + (-1)^k G_k(c-a, c),$$

$$G_k(a,c) = \frac{(-1)^k \cos(c-a)\pi \Gamma(k-c+1)\Gamma(2c-2a)}{2^{2c-2a-1}\Gamma(k+1+c-2a)} \tag{1.10}$$

$$\cdot\, {}_2F_1\left(1-2a, 2c-2a; k+1+c-2a; \frac{1}{2}\right).$$

If a and c are fixed and k is sufficiently large, then the ${}_2F_1$'s in (1.10) are of order unity and

$$\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} R_k(a,c) \approx \frac{(-1)^k \cos(c-a)\pi\Gamma(2c-2a) k^{2a-2c}}{2^{2c-2a-1}} + \frac{\cos a\pi \Gamma(2a) k^{-2a}}{2^{2a-1}}. \tag{1.11}$$

Here we have used the fact that for fixed α and β,

$$\frac{\Gamma(k+\alpha)}{\Gamma(k+\beta)} \approx k^{\alpha-\beta}, \qquad k \text{ large}. \tag{1.12}$$

Again, if z is fixed and k is large, then

$$I_k(z) \approx \frac{(z/2)^k}{k!}. \tag{1.13}$$

It follows that (1.8) converges like

$$\sum_{k=1}^{\infty} \frac{(Ak^{2a-2c} + Bk^{-2a})z^k}{2^{2k}k!}, \tag{1.14}$$

A and B are constants. A similar argument shows that (1.1) converges like

$$\sum_{k=1}^{\infty} \frac{k^{a-c}z^k}{k!}, \tag{1.15}$$

and so (1.8) converges more rapidly than (1.1).

Some general properties of $R_k(a, c)$ are next of interest. The combination of (1.3) and (1.8) shows that

$$R_k(a, c) = (-1)^k R_k(c - a, c); \qquad R_k(a, a) = (-1)^k. \tag{1.16}$$

Further relations follow by applying known contiguous relations of $\Phi(a, c, z)$ to (1.8). For example, since

$$\frac{d}{dz}\Phi(a, c, z) = \frac{az}{c}\Phi(a + 1, c + 1, z), \tag{1.17}$$

it follows that

$$R_k(a + 1, c + 1) = -\frac{c}{4a}[R_{k+1}(a, c) - 2R_k(a, c) + R_{k-1}(a, c)], \tag{1.18}$$

and

$$(a - c + 1)R_k(a, c) - aR_k(a + 1, c) + (c - 1)R_k(a, c - 1) = 0, \tag{1.19}$$

results from a contiguous formula of the form (1.19) with $R_k(a, c)$ replaced by $\Phi(a, c, z)$. To obtain a pure recursion formula for $R_k(a, c)$, apply the differential equation satisfied by $\Phi(a, c, z)$ to (1.8). Employ the difference-differential properties of $I_n(z)$ and equate to zero the coefficient of $I_k(z)$. We get

$$(k + c)R_{k+1}(a, c) = 2(c - 2a)R_k(a, c) + (k - c)R_{k-1}(a, c). \tag{1.20}$$

For certain values of the parameters, the coefficients $R_k(a, c)$ can be expressed in simple form. Thus from (1.10),

$$R_k(\tfrac{1}{2}, c) = \frac{[\Gamma(c)]^2}{\Gamma(c + k)\Gamma(c - k)}, \qquad c \neq 0, -1, -2, \cdots, R(c) > 0. \tag{1.21}$$

This also follows from (1.9) and the known expression for a ${}_2F_1$ with unit argument. If $c = 2a$, then from (1.9) and (1.20), $R_k(a, 2a)$ is zero if k is odd, and

$$R_{2k}(a, 2a) = \frac{(\frac{1}{2} - a)_k}{(\frac{1}{2} + a)_k}, \qquad a \neq -\tfrac{1}{2}, -\tfrac{3}{2}, -\tfrac{5}{2} \cdots \tag{1.22}$$

Since

$$\Phi(\nu + \tfrac{1}{2}, 2\nu + 1, -z) = \Gamma(\nu + 1)(z/4)^{-\nu}e^{-z/2}I_\nu(z/2), \tag{1.23}$$

it follows that

$$I_\nu(z) = (z/2)^\nu \Gamma(\nu+1) \sum_{k=0}^{\infty} \frac{(-1)^k \epsilon_k I_{2k}(z)}{\Gamma(\nu-k+1)\Gamma(\nu+k+1)} \tag{1.24}$$

provided ν is not a negative integer The latter is a special case of a known result [9].

2. The Incomplete Gamma Function. If $c = a + 1$, we write

$$\gamma(a, z) = a^{-1} z^a e^{-z} \Phi(1, 1+a, z) = a^{-1} z^a \Phi(a, a+1, -z), \tag{2.1}$$

$$\gamma(a, z) = \int_0^z e^{-t} t^{a-1}\, dt = \Gamma(a) - \int_z^\infty e^{-t} t^{a-1}\, dt, \qquad R(a) > 0. \tag{2.2}$$

Thus

$$\gamma(a, z) = a^{-1} z^a e^{-z/2} \sum_{k=0}^{\infty} \epsilon_k R_k(a) I_k(z/2), \tag{2.3}$$

$$\begin{aligned} R_k(a) \equiv R_k(a, a+1) &= {}_3F_2(-k, k, a; a+1, \tfrac{1}{2}; 1) \\ &= (-1)^k {}_3F_2(-k, k, 1; a+1, \tfrac{1}{2}; 1) \\ &= \frac{-(-1)^k a}{2(k-a)(k-a+1)}\, {}_3F_1(1-2a, 2; k-a+2; \tfrac{1}{2}) \\ &\quad + a \frac{\cos a\pi \Gamma(2a) k \Gamma(k-a)}{2^{2a-1}\Gamma(k+a+1)}, \end{aligned} \tag{2.4}$$

and

$$(k+a+1)R_{k+1}(a) = 2(1-a)R_k(a) + (k-a-1)R_{k-1}(a), \tag{2.5}$$

$$R_k(a+1) = -\frac{(a+1)}{4a}[R_{k+1}(a) - 2R_k(a) + R_{k-1}(a)]. \tag{2.6}$$

If $a = \frac{1}{2}$, we deduce the following expressions for the error function and related functions.

$$\operatorname{Erf} z = \int_0^z e^{-t^2}\, dt = -z \sum_{k=0}^{\infty} \frac{(-1)^k \epsilon_k e^{-z^2/2} I_k(z^2/2)}{4k^2 - 1}. \tag{2.7}$$

$$e^{-z^2} \operatorname{Erfi} z = e^{-z^2} \int_0^z e^{t^2}\, dt = -z \sum_{k=0}^{\infty} \frac{\epsilon_k e^{-z^2/2} I_k(z^2/2)}{4k^2 - 1}. \tag{2.8}$$

$$(2\pi)^{\frac{1}{2}}[C(z) + iS(z)] = \int_0^z t^{-\frac{1}{2}} e^{it}\, dt = -2z^{\frac{1}{2}} e^{iz/2} \sum_{k=0}^{\infty} \frac{\epsilon_k i^k J_k(z/2)}{4k^2 - 1}. \tag{2.9}$$

A more general expansion for the error function can be found as follows. Now

$$\int_0^{z\sin\theta} e^{-t^2}\, dt = ze^{-z^2/2} \int_0^\theta e^{(z^2/2)\cos 2t} \cos t\, dt, \tag{2.10}$$

and using (1.4), we obtain

$$\operatorname{Erf}(z \sin\theta) = \frac{z}{2} \sum_{k=0}^{\infty} \epsilon_k \left\{ \frac{\sin(2k+1)\theta}{2k+1} + \frac{\sin(2k-1)\theta}{2k-1} \right\} e^{-z^2/2} I_k(z^2/2), \tag{2.11}$$

and from the latter follows the interesting representation

$$(2.12)\qquad \int_0^{\frac{\pi}{2}} \operatorname{Erf}(z\sin\theta)\sin(2k+1)\theta\,d\theta = \frac{\pi z}{4(2k+1)}\,e^{-z^2/2}[I_k(z^2/2)+I_{k+1}(z^2/2)].$$

3. The Exponential Integral. This is a limiting form of the incomplete gamma function. From (2.2), we have

$$(3.1)\qquad \int_z^\infty t^{-1}e^{-t}\,dt = \lim_{a\to 0}\{\Gamma(a)-\gamma(a,z)\}.$$

Another useful form is

$$(3.2)\qquad \int_0^z t^{-1}(1-e^{-t})\,dt = \lim_{a\to 0}\{a^{-1}z^a-\gamma(a,z)\}.$$

Using (1.4) and (2.3), we write

$$(3.3)\qquad e^{z/2}z^{-a}\int_z^\infty e^{-u}u^{a-1}\,du = \left\{\frac{\Gamma(a+1)z^{-a}-1}{a}\right\}e^{z/2}+2\sum_{k=1}^\infty\left\{\frac{1-R_k(a)}{a}\right\}I_k(z/2).$$

Application of L'Hospital's theorem gives

$$(3.4)\qquad \lim_{a\to 0}\left\{\frac{\Gamma(a+1)z^{-a}-1}{a}\right\} = -(\gamma+\ln z)$$

where γ is Euler's constant. Define

$$(3.5)\qquad f_k = \lim_{a\to 0}\frac{1-R_k(a)}{a} = -\sum_{m=1}^{k}\frac{(-1)^m\binom{k}{m}(k)_m}{(\frac{1}{2})_m\,m}.$$

Then

$$(3.6)\qquad \begin{aligned} -Ei(-z)+(\gamma+\ln z) &= \int_z^\infty t^{-1}e^{-t}\,dt+(\gamma+\ln z)\\ &= \int_0^z t^{-1}(1-e^{-t})\,dt = 2\sum_{k=1}^\infty f_k e^{-z/2}I_k(z/2).\end{aligned}$$

In view of (2.5) and (3.5), we have the recurrence formula

$$(3.7)\qquad (k+1)f_{k+1} = 2f_k+(k-1)f_{k-1}+4,$$

and by induction or otherwise, we can prove that

$$(3.8)\qquad \begin{cases} f_{k+1} = f_k+\frac{1}{k}\{1-(-1)^k\}+\frac{1}{k+1}\{1+(-1)^k\}, \qquad f_0=0, f_1=2,\\ f_{2k} = 4\left(1+\frac{1}{3}+\cdots+\frac{1}{2k-1}\right),\\ f_{2k+1} = f_{2k}+\frac{2}{2k+1}, \qquad f_{2k+2} = f_{2k+1}+\frac{2}{2k+1}.\end{cases}$$

Employing the theory of the $\psi(z)$ function, the logarithmic derivative of the

gamma function [10], we find that

$$\lim_{k\to\infty}(f_k - 2\ln 2k) = 2\gamma, \tag{3.9}$$

and so (3.6) converges like

$$\sum_{k=1}^{\infty} u_k z^k, \qquad u_k = (\ln k)/2^{2k}k! \tag{3.10}$$

An alternative form of (3.6) is

$$\begin{cases} \displaystyle\int_0^z t^{-1}(1-e^{-t})\,dt = 2[1 - e^{-z/2}I_0(z/2)] + 4\sum_{k=0}^{\infty} g_{k+2}e^{-z/2}I_{k+2}(z/2), \\ g_k = \tfrac{1}{2}f_k - 1. \end{cases} \tag{3.11}$$

Now

$$Ei(z) = \int_{-\infty}^{z} t^{-1}e^t\,dt = Ei(ze^{\pm i\pi}) \mp i\pi, \tag{3.12}$$

and so

$$\begin{aligned} e^{-z}\int_{-\infty}^{z} t^{-1}e^t\,dt = e^{-z}(\gamma + \ln z) + 2[e^{-z/2}I_0(z/2) - e^{-z}] \\ - 4\sum_{k=0}^{\infty}(-1)^k g_{2k+2}e^{-z/2}I_{k+2}(z/2) \end{aligned} \tag{3.13}$$

We also have

$$\begin{aligned} \int_0^z t^{-1}(1-e^{-it})\,dt &= (\gamma + \ln z) - Ci(z) + iSi(z) \\ &= 2e^{-iz/2}\sum_{k=1}^{\infty} f_k i^k J_k(z/2). \end{aligned} \tag{3.14}$$

4. A Second Expansion and Special Cases. Buchholz [1] has given the expansion

$$\begin{aligned} \Phi(a,c,z) &= e^{z/2}(z/4)^{\frac{1}{2}-a}\Gamma(a-\tfrac{1}{2}) \\ &\cdot \sum_{k=0}^{\infty} \frac{(k+a-\frac{1}{2})(2a-1)_k(2a-c)_k}{k!(c)_k} I_{k+a-\frac{1}{2}}(z/2), \qquad R(a) > 0, \end{aligned} \tag{4.1}$$

which coalesces with (1.8) when $a = \frac{1}{2}$. If a, c and z are fixed, then an analysis similar to (1.11)–(1.14) shows that (4.1) converges like

$$\sum_{k=1}^{\infty} \frac{k^{3a-2c-1/2}z^k}{2^{2k}k!}, \tag{4.2}$$

and from this point of view, the convergence of (1.8) and (4.1) are nearly alike when $R(c-a) > 0$.

We now deduce from (4.1) series expansions for the exponential integral and related functions. The method of proof is akin to that in the previous section. If in (4.1), we put $a = c$, there follows a representation for $e^{z/2}$ since $\Phi(a,a,z) = e^z$. Use this and (4.1) with $c = a + 1$ and z replaced by $-z$. Then after the manner

of (3.3), we have

$$e^{z/2}\int_z^\infty t^{a-1}e^{-t}\,dt = (\pi z)^{1/2}\sum_{k=0}^{\infty} h_k(a)I_{k+a-1/2}(z/2), \tag{4.3}$$

and employing the duplication formula for gamma functions, the elements $h_k(a)$ take the form

$$h_k(a) = \frac{2^{k+2a-1}(k+a-1/2)\Gamma\left(\frac{k-1}{2}+a\right)\Gamma\left(\frac{k}{2}+a\right)}{\Gamma(\frac{1}{2})k!} \cdot\left\{z^{-a} - \frac{(-1)^k}{\Gamma(a-1)(k+a)(k+a-1)}\right\}. \tag{4.4}$$

For $k = 0$ and 1, application of L'Hospital's theorem gives

$$h_0(0) = -\tfrac{1}{2}(\gamma + \ln z), \qquad h_1(0) = h_0(0) + 1. \tag{4.5}$$

Otherwise,

$$h_k(0) = \frac{2k-1}{k(k-1)}, \qquad k > 1. \tag{4.6}$$

Thus

$$\begin{aligned} -Ei(-z) + (\gamma + \ln z) &= \int_z^\infty t^{-1}e^{-t}\,dt + (\gamma + \ln z) = \int_0^z t^{-1}(1-e^{-t})\,dt \\ &= (\pi z)^{1/2}e^{-z/2}I_{1/2}(z/2) + (\pi z)^{1/2}\sum_{k=0}^{\infty}\frac{(2k+3)}{(k+1)(k+2)}e^{-z/2}I_{k+3/2}(z/2), \end{aligned} \tag{4.7}$$

and forms for the related functions follow readily enough after the manner of (3.12)–(3.14). By a familiar argument, (4.7) converges like

$$\sum_{k=1}^{\infty} v_k z^k, \qquad v_k = [2^{2k}k\Gamma(k+\tfrac{1}{2})]^{-1}, \tag{4.8}$$

and so (4.7) converges more rapidly than (3.6). However, the difference is not great since $u_k/v_k \backsim k^{\frac{1}{2}}\ln k$ (see (3.10)).

5. Further Expansions. The following representation is due to Tricomi [2, 3].

$$\Phi(a, c, z) = \Gamma(c)e^{hz}\sum_{k=0}^{\infty} A_k(a, c, h)z^k E_{c+k-1}(-az), \qquad h \geqq 0, \tag{5.1}$$

$$E_\nu(z) = z^{-\nu/2}J_\nu(2z^{\frac{1}{2}}), \qquad E_\nu(-z) = z^{-\nu/2}I_\nu(2z^{\frac{1}{2}}), \tag{5.2}$$

where the coefficients $A_k(a, c, h)$ are defined by the generating function

$$\sum_{k=0}^{\infty} A_k(a, c, h)z^k = e^{-az}[1 + (h-1)z]^{-a}(1+hz)^{a-c}, \qquad |z| < 1, \tag{5.3}$$

and satisfy the recurrence system

$$\begin{aligned} (k+1)A_{k+1} &= [(1-2h)k - hc]A_k \\ &\quad + [a(1-2h) - h(h-1)(c+k-1)]A_{k-1} - h(h-1)A_{k-2}, \\ A_0 = 1, \qquad A_1 &= -hc, \qquad A_2 = \tfrac{1}{2}h^2c(c+1) + a(\tfrac{1}{2} - h). \end{aligned} \tag{5.4}$$

A second expansion given by Tricomi [2, 3] is

$$\Phi(a, c, z) = \Gamma(c)e^{z/2}\sum_{k=0}^{\infty} A_k^*(K, c/2)(z/2)^k E_{c+k-1}(Kz), \qquad K = \frac{c}{2} - a, \tag{5.5}$$

where

$$\sum_{k=0}^{\infty} A_k^*(K, c/2)z^k = e^{2Kz}(1 - z)^{K-c/2}(1 + z)^{-K-c/2}, \qquad |z| < 1, \tag{5.6}$$

$$(k + 1)A_{k+1}^* = (k + c - 1)A_{k-1}^* - 2KA_{k-2}^*,$$

$$A_0^* = 1, \qquad A_1^* = 0, \qquad A_2^* = \frac{c}{2}. \tag{5.7}$$

Both (5.1) and (5.5) are useful to study the behavior of Φ when the parameters are large [11], but are not too suitable for many purposes of computation, since the order and argument of the Bessel functions depend on the parameters a and c.

In (5.1), put $a = 1$, replace c by $a + 1$ and set $h = 0$. Then

$$\gamma(a, z) = \Gamma(a)e^{-z}z^{a/2}\sum_{k=0}^{\infty} e_k(-1)z^{k/2}I_{k+a}(2z^{1/2}), \qquad e_k(-1) = \sum_{m=0}^{k}\frac{(-1)^m}{m!} \tag{5.8}$$

follows from (2.1) and (5.3). The method of proof surrounding (1.11)–(1.14) shows that (5.8) converges like $\sum_{k=0}^{\infty} z^k/\Gamma(k + a + 1)$ which is the same as (1.1) for the above selection of parameters.

6. Further Expansions for the Error and Exponential Integrals. We start with the representation

$$\int_0^z t^{\mu}I_{\nu}(t)\,dt = \frac{z^{\mu}\Gamma\left(\frac{\nu + \mu + 1}{2}\right)}{\Gamma\left(\frac{\nu - \mu + 1}{2}\right)} \cdot\sum_{k=0}^{\infty}(-1)^k\frac{(\nu + 2k + 1)\Gamma\left(\frac{\nu - \mu + 1}{2} + k\right)}{\Gamma\left(\frac{\nu + \mu + 3}{2} + k\right)}I_{\nu+2k+1}(z), \quad R(\mu + \nu + 1) > 0 \tag{6.1}$$

which is readily verified by differentiation. Put $\mu = 0$, and consider the formulas when $\nu = \frac{1}{2}$ and $\nu = -\frac{1}{2}$. Then by subtraction and addition, we get

$$\operatorname{Erf} z = (\pi/2)^{\frac{1}{2}}\sum_{k=1}^{\infty}(-1)^{[k/2]}I_{k-\frac{1}{2}}(z^2). \tag{6.2}$$

$$\operatorname{Erfi} z = (\pi/2)^{\frac{1}{2}}\sum_{k=0}^{\infty}(-1)^{[k/2]}I_{k+\frac{1}{2}}(z^2). \tag{6.3}$$

Here $[k/2]$ is the largest integer, including zero, contained in $k/2$. The latter two equations were also verified by Tricomi [3] by means of the Laplace transform*.

* There is a typographical error in [12] for (6.2). There the summation should start with $n = 1$, not $n = 0$.

Another representation which follows from (5.8) is

$$\text{Erf } z = \tfrac{1}{2}(\pi z)^{\frac{1}{2}} e^{-z^2} \sum_{k=0}^{\infty} e_k (-1) z^k I_{k+\frac{1}{2}}(2z). \tag{6.4}$$

By a familiar argument, the convergence of (2.7) is superior to that of (6.2)–(6.4).

To obtain another expansion for functions related to the exponential integral, it is convenient to again use (6.1) with z replaced by iz. The ensuing formula is of the same type as (6.1) with $I_\nu(z)$ replaced by $J_\nu(z)$ and the factor $(-1)^k$ omitted behind the summation sign. Now put $\mu = -\frac{1}{2}$ and $\nu = \frac{1}{2}$. Then for the sin integral, we have

$$\int_0^z \frac{\sin t}{t}\, dt = \left(\frac{\pi}{2z}\right)^{1/2} \sum_{k=0}^{\infty} \frac{(4k+3)k!}{(\frac{3}{2})_k} J_{2k+3/2}(z) \tag{6.5}$$

Similarly, with $\nu = -\frac{1}{2}$, we can form an expression for

$$\int_0^z t^{\mu-1/2}(1-\cos t)\, dt. \qquad \text{Let } \mu \to -\tfrac{1}{2}; \qquad \text{then}$$

$$\int_0^z t^{-1}(1-\cos t)\, dt = \frac{1}{2}\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \sum_{k=1}^{\infty} \frac{(4k+1)(\frac{1}{2})_k}{k!} \tag{6.6}$$
$$\cdot\{\psi(k+1) - \psi(1) + \psi(k+\tfrac{1}{2}) - \psi(\tfrac{1}{2})\} J_{2k+\frac{1}{2}}(z).$$

Our previous analysis shows that for z fixed and k large, the ratio of the kth term in the expansion (6.6) to the kth term of the Taylor series expansion for the cosine integral is proportional to $k(\ln k)/2^{2k}$. Similarly for (6.5), the ratio is proportional to $k/2^{2k}$. If (3.6) is compared with its corresponding Taylor series representation in like fashion, we again obtain the former ratio. Thus the proper combination of (6.5) and (6.6) gives about the same convergence as (3.6) and (4.7).

As a remark aside, using (6.1), we can produce

$$\begin{aligned}
&\lim_{\mu \to -1} \int_0^z t^\mu [1 - J_0(t)]\, dt \\
&= \int_0^z t^{-1}[1 - J_0(t)]\, dt = (\gamma + \ln z/2) + \int_0^\infty t^{-1} J_0(t)\, dt \\
&= 2z^{-1} \sum_{k=0}^{\infty} (2k+3)[\psi(k+2) - \psi(1)] J_{2k+3}(z) \\
&= 1 - 2z^{-1} J_1(z) + 2z^{-1} \sum_{k=0}^{\infty} (2k+5)[\psi(k+3) - \psi(1) - 1] J_{2k+3}(z).
\end{aligned} \tag{6.7}$$

For tables of this integral when z is real, see [13].

We now derive yet another expression for the exponential integral. It is known [14, 15, 16]* that

* We take this occasion to point out some typographical errors in [15] and [16]. The formula $(\partial J_\nu(z)/\partial \nu)_{\nu=k}$ should read

$$\left.\frac{\partial J_\nu(z)}{\partial \nu}\right|_{\nu=k} = -G_k(z) + \tfrac{1}{2}k! \sum_{n=0}^{k-1} \frac{(\frac{1}{2}z)^{n-k} J_n(z)}{n!(k-n)}, \qquad G_k(z) = -\frac{\pi}{2} Y_k(z).$$

$$Ei(-2z) = (\pi z/2)^{1/2}e^{-z}\left[\frac{\partial I_\nu(z)}{\partial_\nu}\bigg|_{\nu=1/2} + \frac{\partial I_\nu(z)}{\partial_\nu}\bigg|_{\nu=-1/2}\right], \tag{6.8}$$

where $Ei(-z)$ is given by (3.6). To evaluate the partial derivatives in (6.8), we use the expansion [17]

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu)}\sum_{k=0}^{\infty}\frac{(-1)^k z^k I_k(z)}{(\nu+k)k!}, \tag{6.9}$$

and find that

$$\int_0^z t^{-1}(1-e^{-t})\,dt = 2e^{-z/2}\cosh z/2 + zg_1(z) - 2g_2(z), \tag{6.10}$$

$$e^{-z}\int_0^z t^{-1}(e^t-1)\,dt = -2e^{-z/2}\cosh z/2 + zg_1(z) + 2g_2(z), \tag{6.11}$$

where

$$g_m(z) = \sum_{k=0}^{\infty}\frac{(-1)^k(z/4)^k e^{-z/2}I_k(z/2)}{k!(2k+3-2m)^2}, \qquad m = 1,2 \tag{6.12}$$

Of all the exponential integral expressions given, this converges the best.

7. Numerical Examples.*

EXAMPLE 1. Using (1.8) and (1.9), we illustrate the computation of

$$e^{-iz}\Phi(m+1-ia, 2m+2, 2ix) = \sum_{k=0}^{\infty}\epsilon_k R_k i^k J_k(x), \tag{7.1}$$

$$R_k = {}_3F_2(-k, k, m+1+ia; 2m+2, \tfrac{1}{2}; 1). \tag{7.2}$$

Suppose $a = 1$, $m = 2$. Then values of i^kR_k are easily generated using (7.2) and (1.20). We have $R_0 = 1$, $iR_1 = \frac{1}{3}$, $i^2R_2 = \frac{19}{21}$, etc. Using standard tables of Bessel functions, if $x = 1$, seven terms of (7.1) give the six-decimal value 1.451140. If $x = 2$, nine terms give 1.293748. For tables of (7.1), see [18]. If $x = 2$, about 15 terms of the Taylor series are required to achieve six-decimal accuracy.

EXAMPLE 2. We employ (2.7) and (2.8) to compute the error integrals for $z = 2$. Using nine terms, we find Erf 2 = 0.88208 14 and e^{-4}Erfi 2 = 0.30134 04. About 19 terms of the Taylor series expansion are needed for the same accuracy.

EXAMPLE 3. To illustrate computation of the exponential integral, it is sufficient to consider (6.10). If $z = 4$ and we use six terms of each series in (6.12), then the value of the integral (6.10) is 1.96728 94. Similarly, if $z = 8$, only eight terms of each series in (6.12) are needed to yield the value 2.65669 49. For $z = 8$, about 32 terms of the Taylor series expansion would be required to achieve the same accuracy.

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Quadratic Residues and the Distribution of Primes

By Daniel Shanks

1. Introduction. Chebyshev stated, [1], that there were "many more" primes of the form $4m - 1$ than of the form $4m + 1$ and he indicated, [2], at least three senses in which this assertion was to be understood. If the numbers of primes in these two classes which are $\leqq n$ are designated as $\pi_-(n)$ and $\pi_+(n)$ respectively, and if

$$\Delta(n) = \pi_-(n) - \pi_+(n), \tag{1}$$

and

$$t(n) = \frac{\Delta(n)}{\sqrt{n}/\log n}, \tag{2}$$

then Chebyshev stated that for a certain *subsequence*, n_i, of the integers,

$$\lim_{i \to \infty} t(n_i) = +1. \tag{3}$$

Since $\sqrt{n}/\log n \to \infty$, $\pi_-(n)$ could thus be made to *exceed* $\pi_+(n)$ by any amount by an appropriate choice of n.

This theorem (which was one of the above-mentioned three senses) was proven by Phragmén, [3], and later by Landau, [4], [5]. The other two "senses" have never been proven, [5, p. 647]. Since the theorem, (3), is entirely noncommittal as to the behaviour of $\Delta(n)$, or of $t(n)$, for values of n other than those of the subsequence n_i, it is however not a very convincing argument in favor of Chebyshev's "many more" primes assertion. Littlewood, in fact, showed, [6], that there is a constant K such that for two other sequences n_j and n_k we could have

$$t(n_j) > K \log\log\log n_j \tag{4}$$

and

$$t(n_k) < -K \log\log\log n_k\,. \tag{5}$$

Since $\log\log\log n \to \infty$ (with great dignity) these inequalities imply that the right side of (3) may be changed from $+1$ to *any* real number, positive or *negative*, for some sequence, n_l. Because of this Cramér wrote, [7]:

> "Chebyshev's general assertion—'there are many more primes of the form $4n + 3$ than of the form $4n + 1$'—can therefore be true only in a rather limited region."

The purpose of this paper, [8], is to determine the sense in which Chebyshev's general assertion is (nonetheless) probably true, and to identify the nature of the cause which tends to diminish the class of $4n + 1$ primes relative to the $4n - 1$ primes. The problem is generalized to primes, and also to composites, in any arith-

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metic progression. Finally, there are some comments concerning the Littlewood phenomena, $\pi(n) > Li(n)$.

2. The Function $\Delta(n)$. Tables of $\pi_-(n)$ and of $\pi_+(n)$, [9], [10], [11], [12], published prior to the short note of John Leech, [13], were not of sufficient completeness (or, in one case, of sufficient accuracy) to correctly identify Cramér's "rather limited region." Leech computed (on EDSAC) $\Delta(n)$ for $n \leqq 3{,}000{,}000$ and determined that 26,861 was the smallest* n for which $\pi_+(n) > \pi_-(n)$:

$$\Delta(26{,}861) = -1. \tag{6}$$

The author, unaware of Leech's note, carried out a similar computation (on an IBM 704). The following description of $\Delta(n)$ in this range, 1 to $3 \cdot 10^6$, agrees (where it overlaps) with the shorter account of Leech. In Table 1 we divide this range into six regions and in each region give the maximum and minimum values of $\Delta(n)$, the number of intervals during which $\Delta(n)$ remains zero, positive, or negative, and the total number of n's for which $\Delta(n)$ is zero, positive, or negative. The final figures show that $\Delta(n) > 0$ for 99.84% of the $n \leqq 3 \cdot 10^6$. Table 2 identifies some extrema (and some zeros) of $\Delta(n)$.

This detailed description makes it highly plausible that the *predominantly* positive character of $\Delta(n)$ in this range of n is not merely a passing fancy of the integers (as is almost implied by Cramér's remark) but a permanent phenomenon

TABLE 1

Description of $\Delta(n)$ for $1 \leqq n \leqq 3{,}000{,}000$

Regions of n	Max	Min	0 Intervals	+ Int.	− Int.	0 n's	+ n's	− n's
1 − 462	+6	0	5	4	0	10	452	0
463 − 26,832	+31	+1	0	1	0	0	26,370	0
26,833 − 26,926	+1	−1	6	4	1	60	32	2
26,927 − 616,768	+105	+1	0	1	0	0	589,842	0
616,769 − 633,882	+12	−8	101	48	52	1,282	12,428	3,404
633,883 − 3,000 000 +	+256	+1	0	1	0	0	2,366,118	0
						1,352	2,995,242	3,406
						00.05%	99.84%	00.11%

TABLE 2

Some Special Values

$$\begin{aligned} & \Delta(227) = +6 \\ \Delta(461) = & \ \Delta(462) = 0 \\ & \Delta(17{,}971) = +31 \\ \Delta(26{,}861) = & \ \Delta(26{,}862) = -1 \\ & \Delta(359{,}327) = +105 \\ & \Delta(623{,}681) = -8 \\ \Delta(627{,}859) = \cdots = & \ \Delta(627{,}900) = 0 \\ & \Delta(2{,}951{,}071) = +256 \end{aligned}$$

* This first axis crossing of $\Delta(n)$ was discovered independently by J. W. Wrench, Jr.

for which a sufficient number-theoretic cause should be assigned and of which a more precise formulation is desirable.

3. The Function $\tau(n)$. We therefore return to $t(n)$ and seek a better insight into this function. We do not compute $t(n)$ itself, but instead, the somewhat simpler:

$$\tau(n) = \Delta(n)\cdot\sqrt{n}/\pi(n) \tag{7}$$

where $\pi(n)$ is the *total* number of primes $\leqq n$. From the prime number theorem, $\pi(n) \sim n/\log n$, we have

$$\tau(n) \sim t(n). \tag{8}$$

For example, for $n = 2{,}000{,}000$ we have

$$\pi_-(n) = 74{,}516; \quad \pi_+(n) = 74{,}416; \quad \pi(n) = 148{,}933;$$

$$\Delta(n) = 100; \quad \tau(n) = 0.9496; \quad \text{and} \quad t(n) = 1.0259.$$

The function $\tau(1000\,k)$ was computed for the 2000 values $k = 1, 2, \cdots, 2000$. The minimum, the mean, and the maximum of these 2000 values are

$$\begin{aligned} \tau(629{,}000) &= -0.0464, \\ \frac{1}{2000}\sum_{1}^{2000}\tau(1000k) &= 1.0613, \\ \tau(127{,}000) &= 2.0961, \end{aligned} \tag{9}$$

respectively. Only slightly greater extremes would have been obtained had we computed $\tau(n)$ for *all* the $2\cdot 10^6$ n in this range. The distribution of these 2000 values of τ between the extremes was determined by counting the k's for which

$$\frac{m}{16} \leqq \tau(1000k) < \frac{m+1}{16} \qquad (m = -1, 0, 1, \cdots, 33). \tag{10}$$

These counts, $\nu(m)$, are tabulated in Table 3 and plotted in a bar graph in Fig. 1.

The following comments are now in order:

a.) The distribution is roughly normal with a mean of (nearly) $+1$.

b.) The rare cases of $\Delta(n) < 0$ (i.e., $\tau(n) < 0$) are now to be thought of as no more unusual than the equally rare other extreme: $\Delta(n) > 2\pi(n)/\sqrt{n}$ (i.e., $\tau(n) > 2$).

c.) The implication of Littlewood's inequalities, (4) and (5), is that the distribution function has tails of infinite extent. Of course, occurrences of τ far out in a tail will be very rare.

TABLE 3

m	-1	0	1	2	3	4	5	6	7	8	9	10
$\nu(m)$	4	7	9	7	10	4	3	14	11	48	55	80
m	11	12	13	14	15	16	17	18	19	20	21	22
$\nu(m)$	76	114	139	156	139	151	152	136	129	109	107	77
m	23	24	25	26	27	28	29	30	31	32	33	
$\nu(m)$	52	51	55	29	21	19	17	12	5	0	2	

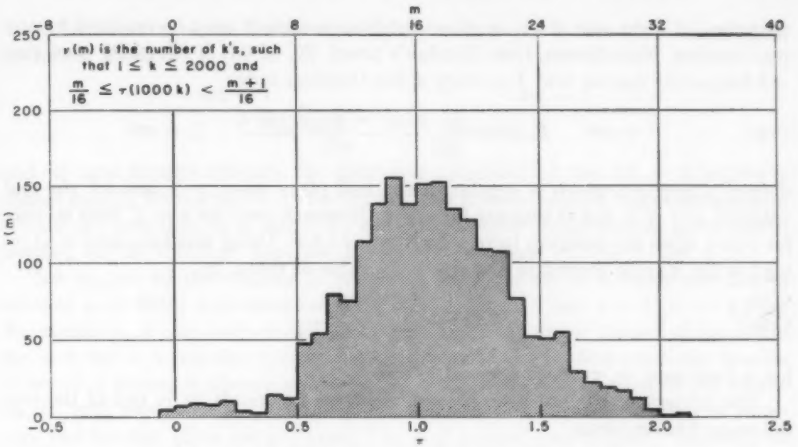

FIG. 1—The distribution of 2000 values of $\tau(1000\ k)$

d.) Chebyshev's function $\sqrt{n}/\log\ n$ (or its equivalent $\pi(n)/\sqrt{n}$) is seen to be an appropriate normalizing factor—it reduces the function $\Delta(n)$ to a function of n with relatively small variation. For example, throughout most of the second million, (1,015,000 to 2,000,000), $\tau(1000k)$ has the following very modest behaviour:

$$\tau(1{,}811{,}000) = 0.426 \leqq \tau(1000k) \leqq 1.610 = \tau(1{,}521{,}000). \tag{11}$$

e.) The function

$$\tau_s = \frac{1}{50s} \sum_{k=1}^{k=50s} \tau(1000k) \qquad (s = 1, 2, \cdots, 40)$$

provides a running history of the mean value of τ, and is found to change very little:

$$\tau_{15} = 1.0222 \leqq \tau_s \leqq 1.1799 = \tau_3 \qquad (s = 1, 2, \cdots, 40). \tag{12}$$

4. A Conjecture. The above discussion suggests that the proper formulation of Chebyshev's "many more" primes assertion is the following

CONJECTURE. The mean value of $\tau(n)$,

$$\frac{1}{N-1} \sum_{2}^{N} \tau(n),$$

has a limit as $N \to \infty$ and this limit equals $+1$;

$$\operatorname*{Lim}_{N\to\infty} \frac{1}{N-1} \sum_{2}^{N} \tau(n) = +1. \tag{13}$$

The conjectured limit, $+1$, may seem a little rash in view of the limited data, (12). However, there is other evidence. The remark made after (5) may suggest

that the $+1$ right side of (3) is of no significance since it may be replaced by any real number. Nonetheless, from Landau's proof, [5], of (3) it is readily seen that $+1$ has a very special role. For every a the Dirichlet series

$$F_a(s) = \sum_{n=2}^{\infty} \frac{\Delta(n) - a\sqrt{n}/\log n}{n^{s+1}} \tag{14}$$

defines a function which is regular in the half plane $\Re(s) > 1$ and for the real values $\frac{1}{2} < s \leqq 1$, but is singular for $s = \frac{1}{2}$. However, only for $a = 1$, that is, only for $F_1(s)$, does the function have a limit as $s \to \frac{1}{2}+$. Using this fact, and $\tau(n) \sim t(n)$, what is to be proved is that the mean value of the a_n in

$$F_1(s) = \sum_{n=2}^{\infty} \frac{a_n}{n^{s+1}\log n} \tag{14a}$$

has a limit as $n \to \infty$ which is equal to zero.

The conjecture has not been proved, either as its stands, or in one of the two following weaker forms:

CONJECTURE. If no zero of

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \cdots$$

has a real part $> \frac{1}{2}$, then (13) is true.

CONJECTURE. If

$$\lim_{N\to\infty} \frac{1}{N-1} \sum_{2}^{N} \tau(n)$$

exists, it equals $+1$.

However, even the original conjecture seems sufficiently likely to be true to merit recording. The closest thing to it, in the literature known to the author, is Theorem 2.351 of Hardy and Littlewood, [6, p. 151]. This, together with their Theorem 2.34 and a third result due to Landau, [7], may be combined and entirely rewritten to read:

THEOREM. If no zero of $L(s)$ has a real part $> \frac{1}{2}$, then there is a positive K such that

$$\frac{1}{N} \sum_{1}^{N} \Delta(n) > K \frac{\sqrt{N}}{\log N}$$

for all sufficiently large N, and conversely.

From this theorem, Abel's lemma on partial summation, [14], and (8), we can easily prove the following

COROLLARY. If no zero of $L(s)$ has a real part $> \frac{1}{2}$, then there is a positive K such that

$$\frac{1}{N-1} \sum_{2}^{N} \tau(n) > K$$

for all sufficiently large N.

We will present further evidence for the conjecture, in section 9 below, after we have determined the number-theoretic cause of the $4m + 1$ prime deficiency.

5. Localization of the Deficiency. If the classes $4m + 1$ and $4m - 1$ are split, modulo 8, we have

$$\overbrace{8m+1 \quad 8m+5}^{4m+1} \qquad \overbrace{8m+3 \quad 8m+7}^{4m-1}$$

and we now inquire whether the deficiency exhibited by the $4m + 1$ primes is shared equally by the $8m + 1$ and the $8m + 5$ primes. Landau's generalization of (3) for any modulus, [5, p. 704–711], and other simpler considerations given below, suggest, on the contrary, that the deficiency is confined entirely to $8m + 1$.

Let $\pi_{a,b}(n)$ be the number of primes of the form $am + b$ which are $\leqq n$. A table of $\pi_{8,b}(1000k)$ was computed for $b = 1, 3, 5$, and 7 and $k = 1, 2, \cdots, 1000$. Examination of this table shows that while the numbers of primes of the forms $8m + 3$, $8m + 5$, and $8m + 7$ are continually jockeying for first place, the number of $8m + 1$ primes is always in last place. Let $f_b(n)$ be the fraction of the *tabular values*, with argument $\leqq n$, for which $am + b$ is in first place. (Two-way or three-way ties for first place are pro-rated $\frac{1}{2}$ or $\frac{1}{3}$ of a point respectively.) Similarly let $l_b(n)$ be the fraction of the tabular values for which $am + b$ is in last place. In the present case, since $8m + 1$ is in last place throughout the entire range of the table, we have for every $k = 1, 2, \cdots, 1000$:

$$f_1(1000k) = l_3(1000k) = l_5(1000k) = l_7(1000k) = 0. \tag{15}$$

A brief summary of $\pi_{8,b}$ and the four other f_b and l_b functions is shown in Table 4. No significant deviations from equality, [15], are to be noted among $8m + 3$, $8m + 5$, and $8m + 7$,— the slightly low $f_5(1{,}000{,}000)$ is compensated by the slightly high $f_5(750{,}000)$. We conclude that the deficiency of the $4m + 1$ primes resides solely in the $8m + 1$ primes. On those rare occasions (regions 3 and 5 of Table 1) when $\pi_+(n) > \pi_-(n)$, we invariably find that $8m + 5$ is in first place with a sufficient lead over $8m + 3$ and $8m + 7$ to overcompensate for the leads which these latter classes hold over $8m + 1$. It is not known whether $8m + 1$ itself can ever take the lead but in view of what is written below about $24m + 1$, this is probable — for sufficiently large n.

TABLE 4

Distribution of the Primes Modulo 8

n	$\pi_{8,1}$	$\pi_{8,3}$	$\pi_{8,5}$	$\pi_{8,7}$	l_1	f_3	f_5	f_7
250,000	5473	5525	5523	5522	1.000	0.446	0.322	0.232
500,000	10334	10418	10397	10388	1.000	0.337	0.344	0.319
750,000	14998	15087	15078	15074	1.000	0.380	0.382	0.237
1,000,000	19552	19653	19623	19669	1.000	0.401	0.287	0.312

6. Distinctions between Modulo 12 and Modulo 10. Alternatively, we could have split $4m + 1$ and $4m - 1$ modulo 12:

$$\overbrace{12m+1 \quad 12m+5}^{4m+1} \qquad \overbrace{12m+7 \quad 12m+11}^{4m-1}$$

TABLE 5
Distribution of the Primes Modulo 12

n	$\pi_{12,1}$	$\pi_{12,5}$	$\pi_{12,7}$	$\pi_{12,11}$	l_1	f_5	f_7	f_{11}
252,000	5504	5567	5564	5566	1.000	0.352	0.420	0.228
504,000	10404	10476	10480	10472	1.000	0.231	0.274	0.495
756,000	15100	15196	15204	15186	1.000	0.396	0.258	0.346
1,008,000	19715	19771	19812	19797	1.000	0.338	0.402	0.260

TABLE 6
Distribution of the Primes Modulo 10

n	$\pi_{10,1}$	$\pi_{10,3}$	$\pi_{10,7}$	$\pi_{10,9}$	l_1	f_3	f_7	l_9
250,000	5495	5520	5517	5510	0.492	0.556	0.444	0.508
500,000	10386	10382	10403	10365	0.555	0.439	0.561	0.445
750,000	15027	15084	15073	15052	0.533	0.445	0.555	0.467
1,000,000	19617	19665	19621	19593	0.502	0.582	0.418	0.498

A table of $\pi_{12,b}(1008k)$ was computed for $b = 1, 5, 7$, and 11 and $k = 1, 2, \cdots, 1000$. (An interval of 1008 was chosen, since its divisibility by 12 simplified the computer program and eliminated even slight inequalities.) The results are similar: the deficiency of $4m + 1$ resides solely in $12m + 1$, and when $\Delta(n) < 0$ we now find $12m + 5$ in the lead. Again,

$$f_1 = l_5 = l_7 = l_{11} = 0 \tag{16}$$

throughout the range computed. Again, the three stronger classes are all equally strong—see Table 5.

In *contrast* now consider the four classes of primes modulo 10, $10m + 1$, $+3$, $+7$, and $+9$. This time we find

$$f_1(1000k) = l_3(1000k) = l_7(1000k) = f_9(1000k) = 0 \tag{17}$$

for $k = 1, 2, \cdots, 1000$ so that for *these* arguments *neither* $10m + 1$ *nor* $10m + 9$ is ever in first place.* We find that $10m \pm 3$ take turns in first place while $10m \pm 1$ take turns in last place. From Table 6 we conclude that this time we have two strong classes (which are equally strong), $10m \pm 3$, and two weak classes (which are equally weak), $10m \pm 1$.

These striking distinctions between modulo 8 and 12 on the one hand and modulo 10 on the other are easily explained in terms of the corresponding modulo multiplication groups, and this explanation provides us with a simple, number—theoretic, sufficient cause.

7. Group Multiplication and Fluctuations. Consider the multiplication table modulo 8 of the four residue classes 1, 3, 5, and 7

* At some *intermediate* arguments, e.g., $n = 135852$, $10m + 1$ is *tied* for first place and similarly, for $n = 969240$, $10m + 7$ is *tied* for last place.

1	3	5	7
3	1	7	5
5	7	1	3
7	5	3	1

and assume that in an interval centered around n there are fewer primes of the form $8m + 1$ than of the other three types. Consider the *excess* primes of the other three classes. Their products with themselves and with each other are composites (of order n^2) whose residue classes are contained in the lower right 3 x 3 box. Now note that these products are distributed into the four residue classes in proportions 3:2:2:2—there is an *extra* product congruent to 1. Assume on the contrary an excess number of primes of the form $8m + 1$. The composites from these primes lie in the upper left 1 x 1 box. Again we find an excess of composites congruent to 1. Similarly, too few or too many primes congruent to 3, 5, or 7 will also lead to an excess number of composites congruent to 1. Briefly, *any fluctuation from equality in the distribution of the primes into the four residue classes will create an excess number of composites and therefore a diminished number of primes congruent to 1 modulo 8.* This idea, that the phenomenon in question is essentially a fluctuation phenomenon (with analogies to the fluctuation phenomena of physics), arose during a discussion of this problem between the author and T. S. Walton. The idea is thus partly due to him.

Since the multiplication group modulo 12 is isomorphic to that modulo 8, the behaviour found modulo 12 is similar. But for 10 we do not have the "vier" group but the cyclic group

1	3	7	9
3	9	1	7
7	1	9	3
9	7	3	1

and this time we find that fluctuations in $10m + 3$ or $10m + 7$ diminish the $10m + 9$ primes while fluctuations in $10m + 1$ or $10m + 9$ diminish the $10m + 1$ primes.

8. Quadratic Residues and the Distribution of Primes. In general, it is readily seen, those residue classes which occur on the principal diagonal are the ones whose numbers of primes are diminished. These are the quadratic residues. We expect then, for any modulus, that the prime race will separate (in the mean) into two races—that between the non-residues, up front, and that between the residues, in the rear. In particular, since $+1$ is the only quadratic residue of 24, the deficiency of $4m + 1$, which was found to reside solely in $8m + 1$ or $12m + 1$, can be further localized to their intersection, $24m + 1$.

For most larger moduli, unlike the unusual 24, there will be many quadratic residues and the deficiency, $\sqrt{n}/\log n$, will be shared among all of them. Thus the separation between the residues and the non-residues will not be as sharp, and extensive interplay is to be expected.

9. Propagation of the Deficiency and Higher Order Effects. The quadratic fluctuation effect just discussed is not merely strong enough to maintain the mean deficiency in the $4m + 1$ primes, but, as we shall see presently, as n increases it becomes *too* strong and we must examine the compensating cubic and higher order effects in order to obtain an accurate picture. We confine ourselves to the modulus 4, although the generalization offers little difficulty. We have seen, (9), that the mean value of τ up to $n = 2 \cdot 10^6$ is nearly one and we wish to show how a mean value equal to one can propagate itself to larger values of n. Our computation, however, is only approximate, and while the result gives further evidence for the truth of (13), it is not a proof of that conjecture.

Let a be a positive integer and let $\pi_+^{(a)}(n)$ be the number of positive integers of the form $4m + 1$ which are $\leqq n$ and which are the products of a (not necessarily distinct) primes. Our previous $\pi_+(n)$ is now $\pi_+^{(1)}(n)$. With a similar definition for $\pi_-^{(a)}(n)$, let

$$\Delta^{(a)}(n) = \pi_-^{(a)}(n) - \pi_+^{(a)}(n), \tag{18}$$

$$\Sigma^{(a)}(n) = \pi_-^{(a)}(n) + \pi_+^{(a)}(n), \tag{19}$$

and

$$\tau^{(a)}(n) = \frac{\Delta^{(a)}(n)}{\Sigma^{(a)}(n)} \sqrt{n}. \tag{20}$$

Note the slight difference between τ and $\tau^{(1)}$—the new denominator, $\Sigma^{(1)}(n) = \pi(n) - 1$, omits the count of the prime 2.

We wish to compute the mean value of $\tau^{(1)}(n)$ and we assume that the mean values of $\tau^{(1)}(x)$ for $x = n^{1/a}$ $(a = 2, 3, 4, \cdots)$ are all equal to 1. It follows that of all the odd primes of order $n^{1/a}$, a fraction equal to $\frac{1}{2}(1 + n^{-1/2a})$ is of the form $4m - 1$ and a fraction equal to $\frac{1}{2}(1 - n^{-1/2a})$ is of the form $4m + 1$. The composites of order n, which are the products of a primes, have as prime factors primes whose (geometric) mean order is $n^{1/a}$. With reference to the multiplication table (mod 4):

$+1$	-1
-1	$+1$

we now find, by induction on a, that among all possible a—fold products of these primes, a fraction equal to $\frac{1}{2}(1 + (-1)^{a+1}n^{-\frac{1}{2}})$ is of the form $4m - 1$ and a fraction equal to $\frac{1}{2}(1 - (-1)^{a+1}n^{-\frac{1}{2}})$ is of the form $4m + 1$. In other words, on the average, we will have, for $a = 2, 3, \cdots$,

$$\bar{\tau}^{(a)}(n) = (-1)^{a+1}. \tag{21}$$

Thus we find a simple and interesting generalization of the Chebyshev phenomenon. There will be, in the mean, an excess of $4m - 1$ composites of odd degree a, and an excess of $4m + 1$ composites of even degree. Further, the mean fractional excess will be $n^{-\frac{1}{2}}$, independent of the degree a.

Now, for any n,

$$0 = \Delta^{(1)}(n) + \Delta^{(2)}(n) + \Delta^{(3)}(n) + \cdots$$

with an error of 0 or 1, and therefore, with the same error,

$$\Delta^{(1)}(n) = -\Delta^{(2)}(n) - \Delta^{(3)}(n) - \cdots. \tag{22}$$

As long as n is not too large, say 1000, most of the composites are of second degree, and we have $\Delta^{(1)}(n) \approx -\Delta^{(2)}(n)$. This, essentially, was our picture in sec. 7 above. But as n increases, the other even degree composites would tend to increase the deficiency while the odd degree composites tend to diminish it. To determine the balance we use the known fact, [5, p. 627], that in any arithmetic progression the number of integers with a odd is asymptotically equal to those with a even. Specifically, for the progression $2m + 1$, we have

$$(1 + \epsilon)\,\Sigma^{(1)}(n) = \Sigma^{(2)}(n) - \Sigma^{(3)}(n) + \Sigma^{(4)}(n) - \cdots,$$

where $\epsilon \to 0$ as $n \to \infty$. Therefore from (20), (21), and (22) we have, in the mean,

$$\Delta^{(1)}(n) = (1 + \epsilon)\,\Sigma^{(1)}(n)/\sqrt{n}.$$

Thus

$$\bar{\tau}^{(1)}(n) = 1 + \epsilon, \tag{23}$$

and a mean value of one propagates itself.

There are two remarks of interest concerning equations (21) and (22).

1.) The sequences $\chi(n)$ and $\lambda(n)$ are defined as follows:

$$\lambda(n) = (-1)^a$$

where a is the number of prime factors of n, and $\chi(n) = 0, 1, 0$, and -1 for numbers of the form $4m$, $4m + 1$, $4m + 2$, and $4m + 3$ respectively. Let

$$b_n = \lambda(n)\chi(n). \tag{24}$$

A generalized Chebyshev assertion now reads: there are "many more" integers m for which $b_m = +1$ than for which $b_m = -1$ and the mean excess is of order $\frac{1}{2}\sqrt{n}$. Since it is known, [5, p. 674], that

$$f(s) = \sum_{n=1}^{\infty} b_n n^{-s} = (1 - 2^{-2s})\zeta(2s)/L(s),$$

it is clear that a function-theoretic formulation of this assertion is concerned with $f(s)$ and with the zeros of $L(s)$.

2.) Let n be sufficiently large such that the number of primes is small compared with the number of composites. Then the terms of like sign in the right side of (22) could be combined so that the right side would be the difference between two nearly equal numbers. We therefore have an analogue of a difficulty in numerical analysis—subtractive loss of significance. In fact, the analogy is very good, since it is known, [15, p. 342], that the normal order of the number of prime divisors of n is log log n and that the number of composites of degree a is given asymptotically by

$$\pi^{(a)}(n) \sim n(\log \log n)^{a-1}/(a-1)! \log n.$$

This means that the numbers of composites of degree a have an approximate Poisson distribution around a mean of log log n and that the equation (22) is comparable to the following equation, which is numerically sensitive when n is large:

$$1 - e^{-\log \log n} = \log \log n - \tfrac{1}{2}(\log \log n)^2 + \cdots .$$

We should therefore expect disturbances in the balance, (23), and more or less random oscillations around this mean. The oscillations should grow as log log n increases and may be expected to show strength when the condition $\Sigma^{(1)} \ll \Sigma^{(2)} \approx \Sigma^{(3)}$ is met. This should occur for log log $n = 2.5+$ or log log log $n \approx 1$.

With these remarks we are led to our final topic.

10. The Littlewood Phenomena. Analogous to Chebyshev's assertion is the erroneous inequality:

$$\pi(n) < Li(n) \tag{24}$$

which was thought to be correct both by Gauss and by Riemann, [7, p. 791, 795]. Again we have the Littlewood counterexample

$$\pi(n) - Li(n) > K \frac{\sqrt{n}}{\log n} \log \log \log n. \tag{25}$$

And again we obtain, this time from Riemann's prime formula, [7, p. 795],

$$Li(n) = \pi(n) + \tfrac{1}{2}\pi(\sqrt{n}) + \cdots , \tag{26}$$

and from $\pi(\sqrt{n}) \sim 2\sqrt{n}/\log n$, the suggestion to define

$$r(n) = \frac{Li(n) - \pi(n)}{\sqrt{n}/\log n}. \tag{27}$$

Or similarly let us define

$$\rho(n) = \frac{Li(n) - \pi(n)}{\pi(n)} \sqrt{n}. \tag{28}$$

Analogy with (13) now suggests the possibility that

$$\lim_{N \to \infty} \frac{1}{N-1} \sum_{2}^{N} \rho(n) = +1 \tag{29}$$

is the proper formulation of the erroneous (24).

If, indeed, the differences $Li(n) - \pi(n)$ and $\pi_-(n) - \pi_+(n)$ are *usually* of the same sign and order of magnitude we should find that

$$\pi^*(n) = \pi(n) + \pi_-(n) - \pi_+(n) = 2\pi_-(n) + 1 \tag{30}$$

agrees with $Li(n)$ much better than $\pi(n)$ does. This is, in fact, the case as is seen in Table 7, [16]. We therefore regard (29) as plausible. It should be investigated.

A final remark concerning the Littlewood phenomena may be of value. It is sometimes said or implied, [17], that (24) is true "im Bereich der Tabellen", where the tables in question include such entries as $n = 10^8$ and $n = 10^9$. But $\pi(n) - Li(n)$ has not been evaluated for all *intermediate* values of n (say for $10^8 < n < 10^9$) and consequently, it is not certain that (24) holds there. In the analogous case of $\Delta(n)$ discussed above, had we computed $\Delta(10{,}000k)$ up to $k = 300$ we would have *always* found $\Delta > 0$ since all the axis crossings of $\Delta(n)$ up to $n = 3.10^6$ occur at intermediate values of n. Further, since the first crossing of $\pi(n) - Li(n)$ may, like Leech's 26861 above, be of very short duration, it could well be missed unless the computing interval in n were rather small. The author knows of no compelling

TABLE 7

n	$\pi(n)$	$\pi^*(n)$	$\pi(n)/Li(n)$	$\pi^*(n)/Li(n)$
10^3	168	175	0.9459	0.9853
$2\cdot10^3$	303	311	0.9625	0.9879
$3\cdot10^3$	430	437	0.9712	0.9870
$4\cdot10^3$	550	561	0.9728	0.9923
$5\cdot10^3$	669	679	0.9777	0.9923
$6\cdot10^3$	783	799	0.9782	0.9982
$7\cdot10^3$	900	915	0.9843	1.0007
$8\cdot10^3$	1007	1015	0.9811	0.9889
$9\cdot10^3$	1117	1125	0.9825	0.9895
10^4	1229	1239	0.9862	0.9943
$2\cdot10^4$	2262	2273	0.9884	0.9932
$3\cdot10^4$	3245	3267	0.9903	0.9970
$4\cdot10^4$	4203	4235	0.9929	1.0005
$5\cdot10^4$	5133	5167	0.9935	1.0001
$6\cdot10^4$	6057	6077	0.9958	0.9990
$7\cdot10^4$	6935	6971	0.9928	0.9980
$8\cdot10^4$	7837	7867	0.9950	0.9988
$9\cdot10^4$	8713	8729	0.9949	0.9968
10^5	9592	9617	0.9961	0.9987
$2\cdot10^5$	17984	18013	0.9971	0.9986
$3\cdot10^5$	25997	26033	0.9965	0.9979
$4\cdot10^5$	33860	33919	0.9981	0.9999
$5\cdot10^5$	41538	41613	0.9983	1.0001
$6\cdot10^5$	49098	49151	0.9985	0.9995
$7\cdot10^5$	56543	56589	0.9982	0.9990
$8\cdot10^5$	63951	64071	0.9986	1.0005
$9\cdot10^5$	71274	71379	0.9988	1.0002
10^6	78498	78645	0.9983	1.0002
$2\cdot10^6$	148933	149033	0.9992	0.9998

reason why the first crossing should occur at anywhere near as large an n as Skewes' fantastic $10^{10^{10^{34}}}$, [18].

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Tables of Abscissas and Weights for Numerical Evaluation of Integrals of the Form $\int_0^\infty e^{-x}x^n f(x)\,dx$

By Philip Rabinowitz and George Weiss

There are many physical problems [1, 2] whose solutions involve integrals of the form

$$\int_0^\infty x^n e^{-x} f(x)\,dx, \tag{1}$$

hence it is desirable to have a table of the weights and abscissas suitable for use in the numerical integration formula of Gaussian type

$$\int_0^\infty x^n e^{-x} f(x)\,dx = \sum_{k=0}^{N} a_{kN}^n f(x_{kN}^n) + E; \tag{2}$$

where

$$E = \frac{N!(N+n)!}{(2N)!} f^{(2n)}(\xi).$$

It is known that if $L_N^n(x)$ is the generalized Laguerre polynomial defined by

$$L_N^n(x) = \sum_{m=0}^{N} \binom{N+n}{N-m} \frac{(-x)^m}{m!} \tag{3}$$

then [3] the abscissas x_{kN}^n appearing in Eq. (2) are the roots of $L_N^n(x) = 0$, while the coefficients a_{kN}^n are given by

$$a_{kN}^n = \frac{1}{x_{kN}^n} \frac{1}{[L_N^{n\prime}(x_{kN}^n)]^2} \tag{4}$$

where L' denotes the derivative of L.

The most extensive tables of a_{kN}^0 and x_{kN}^0 have been given by Salzer and Zucker [4] who tabulated the weights and roots to at most 14 places for $N = 2(1)15$. A very short table of the a_{kN}^n and x_{kN}^n to four places has been given by Burnett [5] for $n = 2, 3, 4$ and $N = 2, 3$. The present tables contain the roots, weights, and the weights multiplied by $\exp(x_{kN}^n)$ for use in the numerical integration formula, Eq. (2). The tables are good to 18 significant figures for $n = 0(1)5$. For $n = 0$ we give the appropriate numbers for $N = 4(4)32$ and for $n > 0$ we give the numbers for $N = 4(4)16$.

As an illustration of a possible use of these tables we cite the calculation of such lattice sums as [6]

$$T_{r,s,t} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n^r m^s}{(m^2+n^2)^t}. \tag{5}$$

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By using the identity

$$z^{-t} = \{\Gamma(t)\}^{-1} \int_0^\infty e^{-zu} u^{t-1}\, du$$

we may convert the expressions for $T_{r,s,t}$ into an integral form

$$T_{r,s,t} = \frac{1}{\Gamma(t)} \int_0^\infty u^{t-1} \sigma_r(u) \sigma_s(u)\, du \tag{6}$$

where

$$\sigma_r(u) = \sum_{n=1}^{\infty} n^r e^{-n^2 u} \tag{7}$$

which has an exponential behavior. Previous tabulations of lattice sums have relied on the use of the Poisson summation formula to find the behavior [6] of $\sigma_r(u)$ for small u or on direct summation of a large number of terms of Eq. (5) by electronic computer [7]. The present tables enable one to evaluate the integral in Eq. (6) by direct numerical integration. To facilitate this method we are now computing tables of theta functions evaluated at the zeros of Laguerre polynomials.

The values in the tables were computed on WEIZAC using double precision floating operations which are accurate to approximately 21 significant figures. The zeros of $L_N^n(x)$ were computed using Newton's method. If $x_{kN}^{n(i)}$ is the ith approximation to the kth root of $L_N^n(x)$, then

$$x_{kN}^{n(i+1)} = x_{kN}^{n(i)} - L_N^n(x_{kN}^{n(i)})/L_N^{n\prime}(x_{kN}^{n(i)}) = x_{kN}^{n(i)} - \Delta \tag{8}$$

where $L_N^n(x)$ is calculated from the recurrence relation [3]

$$\begin{gathered} NL_N^n(x) - (2N - 1 + n - x)L_{N-1}^n(x) + (N - 1 + n)L_{N-2}^n(x) = 0, \\ L_{-1}^n(x) = 0, \qquad L_0^n(x) = 1, \end{gathered} \tag{9}$$

and

$$L_N^{n\prime}(x) = \frac{1}{x}[NL_N^n(x) - (N + n)L_{N-1}^n(x)]. \tag{10}$$

The iteration was terminated when $|\Delta| < 2^{-61}$. For the initial guesses for the iteration, various sources were used. For $n = 0$, $N = 4, 8, 12$ the values given by Salzer and Zucker [4] were used. For $n = 0$, $N = 16, 20$, the values given by Head and Wilson [8] were used. In all other cases, we made use of the following inequality given by Szego [9]

$$\begin{aligned} A &= \frac{j_{kn}^2}{4N + 2(n + 1)} < x_{kN}^n < B \\ &= \frac{(2k + n + 1)^2 + (2k + n + 1)[(2k + n + 1)^2 + \frac{1}{4} - n^2]^{\frac{1}{2}}}{2N + n + 1} \end{aligned} \tag{11}$$

where j_{kn} is the kth zero of $J_n(x)$ and is tabulated in Watson [10] for $n = 0(1)5$, $k = 1(1)40$. The way this was done is as follows. First we determined $\lambda_{kN}^n = (x_{kN}^n - A)/(B - A)$ for $n = 0$, $N = 15, 16$ using the values given by Head and Wilson [8]. From the two graphs of $\lambda_{k,15}^0$ and $\lambda_{k,16}^0$ we estimated that $4(k/(2N-1))^2$ was a good approximation to λ_{kN}^n, independent of n. Hence our initial guess for x_{kN}^n was

$$x_{kN}^{n(0)} = \frac{j_{kn}^2 + 8\left(\frac{k}{2N-1}\right)^2 [(2k+n+1)^2 - \tfrac{1}{2}j_{kn}^2 + (2k+n+1) \cdot \{(2k+n+1)^2 + \tfrac{1}{4} - n^2\}^{\frac{1}{2}}]}{4N+2n+2}. \tag{12}$$

This choice of $x_{kN}^{n(0)}$ was a rather good approximation to x_{kN}^n, differing from it by at most 10% and usually by less than 1%, and always being closer to it than to any other root of $L_N^n(x)$. Convergence to x_{kN}^n occurred in almost every case in at most 10 iterations, and usually in much less. However, in some cases in which $k = N$, Newton's method favored convergence to $x_{N-1,N}^n$. This caused very little trouble for an approximate value for $x_{N,N}^n$ was computed from the relation derived from Eq. (3)

$$\sum_{k=1}^{N} x_{k,N}^n = N(N+n) \tag{13}$$

and used as an initial value in the iteration. In general, this relation together with a second relation derived from Eq. (3)

$$\prod_{k=1} x_{k,N}^n = \frac{(N+n)!}{n!} \tag{14}$$

were used as checks on the computation. They always checked to 18 significant figures. In addition the following quantities were computed by the machine and these also checked to 18 significant figures (all summations for $k = 1, \cdots, N$).

$$\sum a_{kN}^n = n!$$

$$\sum a_{kN}^n x_{kN}^n = (n+1)!$$

$$\sum a_{kN}^n (x_{kN}^n)^7 = (n+7)!$$

$$\sum a_{kN}^n (x_{kN}^n)^{15} = (n+15)! \qquad N \neq 4$$

$$\sum a_{kN}^n (x_{kN}^n)^{23} = (n+23)! \qquad N \neq 4, 8$$

$$\begin{cases} \sum a_{kN}^n (x_{kN}^n)^{31} = (n+31)! & n = 0, 1, 2;\ N \geq 16 \\ \sum a_{kN}^n (x_{kN}^n)^{27} = (n+27)! & n = 3, 4, 5;\ N = 16. \end{cases}$$

The values for $n = 0$, $N = 32$ were not checked by the machine and were only checked by hand using Eq. (13), while for $n = 0$, $N = 28$, only Eqs. (13) and (14) were checked by the computer. This modification was necessary because the values $a_{28,28}^0$, $a_{31,32}^0$, and $a_{32,32}^0$ were out of the range of the double precision floating routine and had to be computed by hand.

TABLE

Zeros of Generalized Laguerre Polynomials	Weights	Weight times Exponential of Zero $= a^n_{kN} \exp(x^n_{kN})$
	$n = 0, N = 4$	
3.22547689619392312(−1)	6.03154104341633602(−1)	8.32739123837889247(−1)
1.74576110115834658(0)	3.57418692437799687(−1)	2.04810243845429682(0)
4.53662029692112798(0)	3.88879085150053843(−2)	3.63114630582151786(0)
9.39507091230113313(0)	5.39294705561327450(−4)	6.48714508440766227(0)
	$n = 0, N = 8$	
1.70279632305101000(−1)	3.69188589341637530(−1)	4.37723410492911373(−1)
9.03701776799379912(−1)	4.18786780814342956(−1)	1.03386934766559764(0)
2.25108662986613069(0)	1.75794986637171806(−1)	1.66970976565877575(0)
4.26670071028765879(0)	3.33434922612156515(−2)	2.37692470175859948(0)
7.04590540239346570(0)	2.79453623522567252(−3)	3.20854091334792628(0)
1.07585160101809952(1)	9.07650877335821310(−5)	4.26857551082513220(0)
1.57406786412780046(1)	8.48574671627253154(−7)	5.81808336867192193(0)
2.28631317368892641(1)	1.04800117487151038(−9)	8.90622621529221140(0)
	$n = 0, N = 12$	
1.15722117358020675(−1)	2.64731371055443190(−1)	2.97209636044410798(−1)
6.11757484515130665(−1)	3.77759275873137982(−1)	6.96462980430597231(−1)
1.51261026977641879(0)	2.44082011319877564(−1)	1.10778139461575215(0)
2.83375133774350723(0)	9.04492222116809307(−2)	1.53846423904282912(0)
4.59922763941834848(0)	2.01023811546340965(−2)	1.99832760627424276(0)
6.84452545311517735(0)	2.66397354186531588(−3)	2.50074576910086687(0)
9.62131684245686704(0)	2.03231592662999392(−4)	3.06532151828238624(0)
1.30060549933063477(1)	8.36505585681979875(−6)	3.72328911078277160(0)
1.71168551874622557(1)	1.66849387654091026(−7)	4.52981402998173506(0)
2.21510903793970057(1)	1.34239103051500415(−9)	5.59725846183532109(0)
2.84879672509840003(1)	3.06160163503502078(−12)	7.21299546092587700(0)
3.70991210444669203(1)	8.14807746742624168(−16)	1.05438374619100811(1)
	$n = 0, N = 16$	
8.76494104789278403(−2)	2.06151714957800994(−1)	2.25036314864247252(−1)
4.62696328915080832(−1)	3.31057854950884166(−1)	5.25836052762342454(−1)
1.14105777483122686(0)	2.65795777644214153(−1)	8.31961391687087088(−1)
2.12928364509838062(0)	1.36296934296377540(−1)	1.14609924096375170(0)
3.43708663389320665(0)	4.73289286941252190(−2)	1.47175131696680859(0)
5.07801861454976791(0)	1.12999000803394532(−2)	1.81313468738134816(0)
7.07033853504823413(0)	1.84907094352631086(−3)	2.17551751969460745(0)
9.43831433639193878(0)	2.04271915308278460(−4)	2.56576275016502921(0)
1.22142233688661587(1)	1.48445868739812988(−5)	2.99321508637137516(0)
1.54415273687816171(1)	6.82831933087119956(−7)	3.47123448310209029(0)
1.91801568567531349(1)	1.88102484107967321(−8)	4.02004408644466886(0)
2.35159056939919685(1)	2.86235024297388162(−10)	4.67251660773285426(0)
2.85787297428821404(1)	2.12707903322410297(−12)	5.48742065798615247(0)
3.45833987022866258(1)	6.29796700251786779(−15)	6.58536123328921366(0)
4.19404526476883326(1)	5.05047370003551282(−18)	8.27635798436423448(0)
5.17011603395433184(1)	4.16146237037285519(−22)	1.18242775516584348(1)
	$n = 0, N = 20$	
7.05398896919887534(−2)	1.68746801851113862(−1)	1.81080062418989255(−1)
3.72126818001611444(−1)	2.91254362006068282(−1)	4.22556767878563975(−1)
9.16582102483273565(−1)	2.66686102867001289(−1)	6.66909546701848151(−1)
1.70730653102834388(0)	1.66002453269506840(−1)	9.15352372783073673(−1)
2.74919925530943213(0)	7.48260646687923705(−2)	1.16953970719554597(0)
4.04892531385088692(0)	2.49644173092832211(−2)	1.43135498592820599(0)
5.61517497086161651(0)	6.20255084457223685(−3)	1.70298113798502272(0)
7.45901745367106331(0)	1.14496238647690824(−3)	1.98701589079274721(0)
9.59439286958109677(0)	1.55741773027811975(−4)	2.28663578125343079(0)
1.20388025469643163(1)	1.54014408652249157(−5)	2.60583472755383333(0)
1.48142934426307400(1)	1.08648636651798235(−6)	2.94978373421395086(0)
1.79488955205193760(1)	5.33012090955671475(−8)	3.32539578200931955(0)
2.14787882402850110(1)	1.75798117905058200(−9)	3.74225547058981092(0)
2.54517027931869055(1)	3.72550240251232087(−11)	4.21423671025188042(0)
2.99325546317006120(1)	4.76752925178190052(−13)	4.76251846149020929(0)
3.50134342404790000(1)	3.37284424336243841(−15)	5.42172604424557430(0)
4.08330570567285711(1)	1.15501433950039883(−17)	6.25401235693242129(0)
4.76199940473465021(1)	1.53952214058234355(−20)	7.38731438905443455(0)
5.58107957500638989(1)	5.28644272556915783(−24)	9.15132873098747960(0)
6.65244165256157538(1)	1.65645661249902330(−28)	1.28933886459399966(1)

TABLE—*Continued*

Zeros of Generalized Laguerre Polynomials	Weights	Weight times Exponential of Zero = $\sigma^{\alpha}_{kN} \exp(x^{\alpha}_{kN})$
	$n = 0, N = 24$	
5.90198521815079770(−2)	1.42811973334781851(−1)	1.51494412859509452(−1)
3.11239146198483727(−1)	2.58774107517423903(−1)	3.53256582529923847(−1)
7.66096905545936646(−1)	2.58806707272869802(−1)	5.56784563288152601(−1)
1.42559759080361309(0)	1.83322688977778025(−1)	7.62685317697309109(−1)
2.29256205863219029(0)	9.81662726299188922(−2)	9.71872632246547580(−1)
3.37077426420899772(0)	4.07324781514086460(−2)	1.18535789303780109(0)
4.66508370346717079(0)	1.32260194051201567(−2)	1.40426562728441853(0)
6.18153511873676541(0)	3.36934905847830355(−3)	1.62986861575704148(0)
7.92753924717215218(0)	6.72162564093547890(−4)	1.86363505533207295(0)
9.91209801507770602(0)	1.04461214659275180(−4)	2.10729115108148019(0)
1.21461027117297656(1)	1.25447219779933332(−5)	2.36290589104193500(0)
1.46427322895966743(1)	1.15131581273727992(−6)	2.63300875316385675(0)
1.74179926465089787(1)	7.96081295913363026(−8)	2.92075757972772467(0)
2.04914600826164247(1)	4.07285898754999966(−9)	3.23018513349235362(0)
2.38873298481697332(1)	1.50700822629258492(−10)	3.56657337736875657(0)
2.76359371743327174(1)	3.91773651505845138(−12)	3.93704375545516030(0)
3.17760413523747233(1)	6.89418105295808569(−14)	4.35153118886351197(0)
3.63584058016516217(1)	7.81980038245944847(−16)	4.82448185489803577(0)
4.14517204848707670(1)	5.35018881301003760(−18)	5.37802207978918232(0)
4.71531064451563230(1)	2.01051746455550347(−20)	6.04841781261996523(0)
5.36085745446950698(1)	3.60576586455295904(−23)	6.90089835218049651(0)
6.10585314472187616(1)	2.45181884587840269(−26)	8.06996515614695609(0)
6.99622400351050304(1)	4.08830159368065782(−30)	9.90279331948422546(0)
8.14982792339488854(1)	5.57534578832835675(−34)	1.38205320947920057(1)
	$n = 0, N = 28$	
5.07346248498738876(−2)	1.23778843954286428(−1)	1.30220749260603611(−1)
2.67487268640741084(−1)	2.32279276000901161(−1)	3.03513987299129915(−1)
6.58136628354791519(−1)	2.47511896036477212(−1)	4.77992610899299887(−1)
1.22397180838490772(0)	1.92307113132382827(−1)	6.53972597062647847(−1)
1.96676761247377770(0)	1.16405361721130006(−1)	8.32011455708276465(−1)
2.88888332603032189(0)	5.63459053644773065(−2)	1.01271781273780542(0)
3.99331165925011414(0)	2.20663643262588079(−2)	1.19675156048109225(0)
5.28373606284344256(0)	7.02588763558386773(−3)	1.38483686392073190(0)
6.76460340424350515(0)	1.82060789269585487(−3)	1.57777965578712887(0)
8.44121632827132449(0)	3.83344303857123177(−4)	1.77648966944358961(0)
1.03198504629932601(1)	6.53508708069439831(−5)	1.98200839256100634(0)
1.24079034144606717(1)	8.97136205341076834(−6)	2.19554512993098825(0)
1.47140851641357488(1)	9.84701225624928887(−7)	2.41852439028379250(0)
1.72486634156080563(1)	8.56407585267304245(−8)	2.65264936115550918(0)
2.00237833299517127(1)	5.83683876313834429(−9)	2.89998868599490885(0)
2.30538901350302960(1)	3.07563887784230228(−10)	3.16309775591259801(0)
2.63562973744013176(1)	1.23259095272442282(−11)	3.44519246556198147(0)
2.99519668335961821(1)	3.68217367410831200(−13)	3.75040515240709831(0)
3.38666055165844592(1)	7.99879057596890965(−15)	4.08417387436021254(0)
3.81322544101946468(1)	1.22492250032408341(−16)	4.43585716433002825(0)
4.27896723707725763(1)	1.27112429503067374(−18)	4.86974939776545707(0)
4.78920716336227437(1)	8.48859336768654320(−21)	5.34685205218439829(0)
5.35112979596642942(1)	3.40245537942551185(−23)	5.90818195463222266(0)
5.97487960846412408(1)	7.42015658886748513(−26)	6.59151903072765714(0)
6.67569772839064696(1)	7.60041320580173769(−29)	7.46490124681590107(0)
7.47867785233391618(1)	2.87391031794039581(−32)	8.66878601010575288(0)
8.43178371072270431(1)	2.54182290388931800(−36)	1.05661475342792109(1)
9.65824206275273191(1)	1.66137587802903396(−41)	1.46446915172544714(1)
	$n = 0, N = 32$	
4.44893658332670184(−2)	1.09218341952384971(−1)	1.14187105768104849(−1)
2.34526109519618537(−1)	2.10443107938813234(−1)	2.66065216897615217(−1)
5.76884629301886426(−1)	2.35213229669848005(−1)	4.18793137324852994(−1)
1.07244875381781763(0)	1.95903335972881043(−1)	5.72532846499804707(−1)
1.72240877644464544(0)	1.29983786286071761(−1)	7.27648788380971311(−1)
2.52833670642579488(0)	7.05786238657174415(−2)	8.84536719340249717(−1)
3.49221327302199449(0)	3.17609125091750703(−2)	1.04361887589207697(0)
4.61645676974976739(0)	1.19182148348385571(−2)	1.20534927415235258(0)
5.90395850417424395(0)	3.73881629461152479(−3)	1.37022133852178119(0)
7.35812673318624111(0)	9.80803306614955132(−4)	1.53877725646864475(0)
8.98294092421259610(0)	2.14864918801364188(−4)	1.71161935268645726(0)
1.07830186325399721(1)	3.92034196798794720(−5)	1.88942406344948410(0)
1.27636979867427251(1)	5.93454161286863288(−6)	2.07295034024653367(0)

TABLE—*Continued*

Zeros of Generalized Laguerre Polynomials	Weights	Weight times Exponential of Zero = $a^n_{kN} \exp(x^n_{kN})$
1.49311397555225573 (1)	7.41640457866755222 (−7)	2.26310663399696361 (0)
1.72924543367153148 (1)	7.60456787912078148 (−8)	2.46088907248823613 (0)
1.98558609403360547 (1)	6.35060222662580674 (−9)	2.66750812639711715 (0)
2.26308890131967745 (1)	4.28138297104092888 (−10)	2.88439209292204179 (0)
2.56286360224592478 (1)	2.30589949189133608 (−11)	3.1132613270395861 7 (0)
2.88621018163234747 (1)	9.79937928872709406 (−13)	3.35621769259580256 (0)
3.23466291539647370 (1)	3.23780165772926646 (−14)	3.61586985648426880 (0)
3.61004948057519738 (1)	8.17182344342071943 (−16)	3.89551304494854956 (0)
4.01457197715394415 (1)	1.54213383339382337 (−17)	4.19939410471158549 (0)
4.45092079957549380 (1)	2.11979229016361861 (−19)	4.53311497853436176 (0)
4.92243949873086392 (1)	2.05442967378804543 (−21)	4.90427028761124484 (0)
5.43337213333969073 (1)	1.34698258663739516 (−23)	5.32350097202366611 (0)
5.98925091621340182 (1)	5.66129413039735937 (−26)	5.80633321423362136 (0)
6.5975377287935052 8 (1)	1.41856054546303691 (−28)	6.37661467415965254 (0)
7.26876280906627086 (1)	1.91337549445422431 (−31)	7.07352658070724211 (0)
8.01874697791352311 (1)	1.19224876009822236 (−34)	7.96769350929590065 (0)
8.87353404178923987 (1)	2.67151121924013699 (−38)	9.20504033127818968 (0)
9.88295428682839726 (1)	1.33861694210625628 (−42)	1.11630130907678735 (1)
1.11751398097937695 (2)	4.51053619389897424 (−48)	1.53901804152606398 (1)
	$n = 1, N = 4$	
7.43291927981431435 (−1)	4.468705932187763 10 (−1)	9.39700286292286231 (−1)
2.57163500764627847 (0)	4.77635772363868313 (−1)	6.25091709508283381 (0)
5.73117875168909963 (0)	7.41777847310521364 (−2)	2.28714219375495781 (1)
1.09538943126831905 (1)	1.31584968630324014 (−3)	7.52353830776884715 (1)
	$n = 1, N = 8$	
4.09383573203185153 (−1)	1.87632541405723668 (−1)	2.82553823014304850 (−1)
1.38496318480313988 (0)	4.38985360731142368 (−1)	1.75360553036282958 (0)
2.95625455616886207 (0)	2.89996070781313351 (−1)	5.57541442577641522 (0)
5.18194310104007138 (0)	7.51413846166973434 (−2)	1.33773006498366668 (1)
8.16170968814581733 (0)	7.93264664870734492 (−3)	2.77973624568047784 (1)
1.20700551268371548 (1)	3.08642136813304225 (−4)	5.38782591536632858 (1)
1.72497355261489875 (1)	3.34895820979709614 (−6)	1.03842391519935203 (2)
2.45859552436527819 (1)	4.72139282319322132 (−9)	2.24706083439768244 (2)
	$n = 1, N = 12$	
2.82858348239914112 (−1)	1.01453765665592267 (−1)	1.34620745484392543 (−1)
9.52326041364613503 (−1)	3.17699483362918101 (−1)	8.23391640164269559 (−1)
2.01649213857768073 (0)	3.39986271374452795 (−1)	2.55395234400434764 (0)
3.49235406977802504 (0)	1.79532599439199819 (−1)	5.90001913843717294 (0)
5.40549102001572284 (0)	5.20319390353283915 (−2)	1.15836368199363577 (1)
7.79281394041241498 (0)	8.49161743576890271 (−3)	2.05762635029790113 (1)
1.07073886889890908 (1)	7.67135797278597991 (−4)	3.42793100498839410 (1)
1.42271523637899807 (1)	3.63620912038435440 (−5)	5.48810758737070439 (1)
1.84719966342299991 (1)	8.18456700779946096 (−7)	8.61552836575302900 (1)
2.36417837524008953 (1)	7.32315410040323253 (−9)	1.35579615855293272 (2)
3.01200586261063443 (1)	1.83970243442130593 (−11)	2.21678124758432626 (2)
3.88892843760953185 (1)	5.37762852227037045 (−15)	4.16862096414656053 (2)
	$n = 1, N = 16$	
2.16140305239452255 (−1)	6.32773328795394308 (−2)	7.85446678289795171 (−2)
7.26388243251803954 (−1)	2.31090461520719732 (−1)	4.77802508713723198 (−1)
1.53359316037354132 (0)	3.16933542163999097 (−1)	1.46892374608307634 (0)
2.64497099861191096 (0)	2.37894217875244607 (−1)	3.35027298906274298 (0)
4.07097816088019065 (0)	1.10272743359247882 (−1)	6.46355621447558663 (0)
5.82585551510560455 (0)	3.30895883563969837 (−2)	1.12157486249617326 (1)
7.92850418530666709 (0)	6.52571640779246630 (−3)	1.81106406690387663 (1)
1.04038082899510393 (1)	8.42711436792047207 (−4)	2.77968394748499728 (1)
1.32846610707070382 (1)	6.99619095556818742 (−5)	4.11449606829674792 (1)
1.66151732168666126 (1)	3.61208604213355474 (−6)	5.93794445636475115 (1)
2.04560060200272169 (1)	1.10154125147424469 (−7)	8.43201366918135643 (1)
2.48938470253519108 (1)	1.83568251273250421 (−9)	1.18866087257456150 (2)
3.00598629202025761 (1)	1.48148886588507814 (−11)	1.68085784287149817 (2)
3.61706945436791780 (1)	4.73484535903961320 (−14)	2.42124561031249231 (2)
4.36403651841768370 (1)	4.08414605500225599 (−17)	3.66328790805694462 (2)
5.35291511602684204 (1)	3.62395496247891406 (−21)	6.40615970837448162 (2)

TABLE—*Continued*

Zeros of Generalized Laguerre Polynomials	Weights	Weight times Exponential of Zero = $a^n_{kN} \exp(x^n_{kN})$
	$n = 2, N = 4$	
1.22676326350030207 (0)	7.25524997698654378 (−1)	2.47416635265254532 (0)
3.41250735869694597 (0)	1.06342291979194578 (0)	3.22655960312239398 (1)
6.90269260585161340 (0)	2.06696131028353551 (−1)	2.05652340494177747 (2)
1.24580367719511386 (1)	4.35457929379748887 (−3)	1.12047669645817929 (3)
	$n = 2, N = 8$	
6.99330392297772446 (−1)	2.24797751904342970 (−1)	4.52384060238173233 (−1)
1.89881649533754637 (0)	7.99530958902891337 (−1)	5.33925681844090349 (0)
3.67761476834163509 (0)	7.13670609157191155 (−1)	2.82270577764238765 (1)
6.09929454816086345 (0)	2.31541145990131662 (−1)	1.03161617804640828 (2)
9.26742581328238619 (0)	2.91357867305206514 (−2)	3.08474054427306454 (2)
1.33607382722601106 (1)	1.30770014757761649 (−3)	8.29856650024944985 (2)
1.87281386688430816 (1)	1.60219339506628624 (−5)	2.17892467611196491 (3)
2.62686410414766043 (1)	2.52333939442792311 (−8)	6.46102645632067593 (3)
	$n = 2, N = 12$	
4.90239109177454061 (−1)	9.51482251459639793 (−2)	1.55349132168384772 (−1)
1.32377645578306233 (0)	4.77196653262217159 (−1)	1.79310697194985889 (0)
2.54213223523224643 (0)	7.20466902576669237 (−1)	9.15478262411010895 (0)
4.16451935324397066 (0)	4.93826609735938395 (−1)	3.17835397356631661 (1)
6.21800163077006985 (0)	1.75757334769862195 (−1)	8.81773930663432268 (1)
8.74078952492664456 (0)	3.38929300594179609 (−2)	2.11926747809213372 (2)
1.17872181154281359 (1)	3.51882418847753131 (−3)	4.62936424650790635 (2)
1.54365948600615973 (1)	1.87787479831631979 (−4)	9.49936530531789858 (2)
1.98104815433429476 (1)	4.68668021445018989 (−6)	1.88125704585776709 (3)
2.51111194768538[illegible] (1)	4.59806360065300972 (−8)	3.69994780357075735 (3)
3.17262495017079404 (1)	1.25743502885190380 (−10)	7.55128122489510488 (3)
4.06488781934720538 (1)	3.99508663346059607 (−14)	1.79932534090427829 (4)
	$n = 2, N = 16$	
3.77613508344740734 (−1)	4.86064094670787025 (−2)	7.09069825088867147 (−2)
1.01749195760256570 (0)	2.93347390190440231 (−1)	8.11471686778248555 (−1)
1.94775802042424383 (0)	5.83219363383550989 (−1)	4.09008651869529020 (0)
3.17692724488986868 (0)	5.81874148596173479 (−1)	1.39492556822190547 (1)
4.71624006979179562 (0)	3.38180537473790217 (−1)	3.77907618869590957 (1)
6.58058826577491250 (0)	1.22105963944979891 (−1)	8.80339214718968970 (1)
8.78946527064707413 (0)	2.81146258006637302 (−2)	1.84564671246403063 (2)
1.13683230828333189 (1)	4.14314919248225670 (−3)	3.58534172771350011 (2)
1.43506267274370444 (1)	3.85648533767438300 (−4)	6.58551410235728057 (2)
1.77810957248416460 (1)	2.20158005631090712 (−5)	1.16135832614353863 (3)
2.17210847965713089 (1)	7.34236243815651751 (−7)	1.99151272949358401 (3)
2.62581386751110680 (1)	1.32646044204804137 (−8)	3.36092665129919429 (3)
3.15245960042758187 (1)	1.15266648290842947 (−10)	5.65798450641837801 (3)
3.77389210025289391 (1)	3.94706915124608697 (−13)	9.68455967415686438 (3)
4.53185461100898426 (1)	3.63797825636053360 (−16)	1.74765056820267912 (4)
5.53325835388358122 (1)	3.45457612313612400 (−20)	3.70706727320259718 (4)
	$n = 3, N = 4$	
1.75552164718549145 (0)	1.86033407414649995 (0)	1.07647588421845741 (1)
4.26560565665682345 (0)	3.35691019028920320 (0)	2.39037661575137080 (2)
8.05794068313800185 (0)	7.64453972843517610 (−1)	2.41474077829377203 (3)
1.39209318040196832 (1)	1.83209339810621145 (−2)	2.03578296138366883 (4)
	$n = 3, N = 8$	
1.02996168735087822 (0)	4.47659268688580562 (−1)	1.25387504291512878 (0)
2.43991423401425973 (0)	2.12957515366677356 (0)	2.44306070979065246 (1)
4.41318676383851689 (0)	2.37008143827449573 (0)	1.95607691717435238 (2)
7.01921044285466378 (0)	9.13728082731325425 (−1)	1.02145993175804680 (3)
1.03653585615970610 (1)	1.32197368598833373 (−1)	4.19605176057880639 (3)
1.46343284911153273 (1)	6.66810091481878879 (−3)	1.51220362170062521 (4)
2.01808472740399451 (1)	9.04305893787150415 (−5)	5.25709678492352907 (4)
2.79171925451893479 (1)	1.56535793848369390 (−7)	2.08399336850716076 (5)
	$n = 3, N = 12$	
7.31333453524153573 (−1)	1.52730239391842707 (−1)	3.17350448025970021 (−1)
1.72200587771537790 (0)	1.07163463296299004 (0)	5.99659048258816783 (0)
3.08711820886747629 (0)	2.09369597825120887 (0)	4.58809098751416806 (1)
4.84907183643667852 (0)	1.75927719487292856 (0)	2.24522264593643932 (2)
7.03654192274923744 (0)	7.38876099420103765 (−1)	8.40432710605229296 (2)
9.68900419917576858 (0)	1.63538474605744857 (−1)	2.63937440838205532 (3)
1.28619916279440914 (1)	1.90887802863997279 (−2)	7.35648187888716898 (3)

TABLE—*Continued*

Zeros of Generalized Laguerre Polynomials	Weights	Weight times Exponential of Zero $= a_{kN}^{n} \exp(x_{kN}^{n})$
1.66361029842912300(1)	1.12749487761940805(−3)	1.89270227412433568(4)
2.11344882654905403(1)	3.07769783328946458(−5)	4.64320234766113947(4)
2.65616764973530448(1)	3.27387146853583194(−7)	1.12370201079018605(5)
3.33094793553748032(1)	9.65351819752401242(−10)	2.82363592457219416(5)
4.23811857710775980(1)	3.30506360143886766(−13)	8.41579093496417164(5)
	$n = 3, N = 16$	
5.67443458991574145(−1)	6.50981121009449315(−2)	1.14816936994675722(−1)
1.33290773275989334(0)	5.65273224236402383(−1)	2.14354639377120874(0)
2.38148248007005849(0)	1.48989313857906502(0)	1.61220335260503062(1)
3.72382664209342697(0)	1.86124704489653486(0)	7.70976931131275901(1)
5.37212395216187100(0)	1.30047554848272360(0)	2.80018019446226477(2)
7.34193666282613523 4(0)	5.47819646856915096(−1)	8.45668526756508810(2)
9.65333213726122905(0)	1.43843043208105701(−1)	2.24015331269839424(3)
1.23323014070182379(1)	2.37498472421622760(−2)	5.38903830867048069(3)
1.54128500654077155(1)	2.44244469350610863(−3)	1.20653652088452424(4)
1.89402755758602723(1)	1.52340599558850768(−4)	2.56137300941415792(4)
2.29765957156018692(1)	5.50126530501356105(−6)	5.23686431947102649(4)
2.76101814474260749(1)	1.06842629643597444(−7)	1.04639104770985088(5)
3.29745092032584955(1)	9.92524335897222214(−10)	2.07677073163230993(5)
3.92898232533641479(1)	3.61863342780076823(−12)	4.18695241024345828(5)
4.69768962767103126(1)	3.54373278073215328(−15)	8.93856486144031606(5)
5.71135140237534688(1)	3.58180355287779219(−19)	2.28131757123540439(6)
	$n = 4, N = 4$	
2.31915524835569955(0)	6.57222031198722368(0)	6.68203029533717859(1)
5.12867199360117916(0)	1.37936031844206074(1)	2.32826105813133801(3)
9.20089134897243093(0)	3.54028469036795246(0)	3.50699000772320933(4)
1.53512814090706903(1)	9.38918132242164933(−2)	4.36118522676941444(5)
	$n = 4, N = 8$	
1.39445874535841327(0)	1.27551104534833669(0)	5.14386973811936581(0)
3.00412262031589098(0)	7.55971394358976204(0)	1.52468188096643765(2)
5.16118127238140954(0)	9.98706281698755810(0)	1.74144726764228389(3)
7.94175644134279969(0)	4.42018619472331421(0)	1.24308710943862802(4)
1.14570496332457916(1)	7.17052700563446937(−1)	6.78085582369019424(4)
1.58935676714650236(1)	3.98824021635586082(−2)	3.18617808357226932(5)
2.16116099678724373(1)	5.89788576623485837(−4)	1.43383544036118813(6)
2.95362536480182341(1)	1.10804739994015747(−6)	7.44715365181727399(6)
	$n = 4, N = 12$	
1.00157108495383152(0)	3.59159548108807725(−1)	9.77831924001506431(−1)
2.14370613511225703(0)	3.26582820687688745(0)	2.78607678171097753(1)
3.64933321432680176(0)	7.82396142854537168(0)	3.00823651354474138(2)
5.54486813974946731(0)	7.75650688373216868(0)	1.98505168139673751(3)
7.86076259241096344(0)	3.73730972528196261(0)	9.69270306643015021(3)
1.06377654649934495(1)	9.29328933322615071(−1)	3.87339956748239464(4)
1.39325683951155931(1)	1.19937171658365840(−1)	1.34831503047832052(5)
1.78270063100842237(1)	7.73699187701495750(−3)	4.27309561007931643(5)
2.24457538203894399(1)	2.28490954516935571(−4)	1.27919289421063792(6)
2.79955513560437942(1)	2.61140180598097782(−6)	3.75999417306024579(6)
3.48721707324949280(1)	8.23746583872488555(−9)	1.14970110208989482(7)
4.40889427543252506(1)	3.01723739243414224(−12)	4.23832281907530249(7)
	$n = 4, N = 16$	
7.82339164085635910(−1)	1.29739702663052232(−1)	2.83686376560111066(−1)
1.67007183671874001(0)	1.49748665808338508(0)	7.95547187624795022(0)
2.83292171947257737(0)	4.94612614281207374(0)	8.40596269993120727(1)
4.28441836819021891(0)	7.43026705086785796(0)	5.39142636980744192(2)
6.03786871096889479(0)	6.05687666231872829(0)	2.53782571556701110(3)
8.10954933413974021(0)	2.90916131357755320(0)	9.67609960197269259(3)
1.05201017048502218(1)	8.55584651819837666(−1)	3.17018863503759819(4)
1.32960399123563184(1)	1.56009598652813629(−1)	9.28000231797978033(4)
1.64718887086961996(1)	1.75191359650891542(−2)	2.49553149851783729(5)
2.00934999730530394(1)	1.18227257300007126(−3)	6.29816124368594188(5)
2.42235299428062549(1)	4.58506311659349061(−5)	1.51876543553876336(6)
2.89511495806519383(1)	9.50616079827638761(−7)	3.55901373314319768(6)
3.44109430408020216(1)	9.38309759503402079(−9)	8.25712602731401404(6)
4.08248952713820481(1)	3.62281000770432509(−11)	1.94568396347320976(7)
4.86170549734729113(1)	3.75119190628700090(−14)	4.87851881245905145(7)
5.88737277583532394(1)	4.01670675392835967(−18)	1.48732275042273460(8)

TABLE—*Concluded*

Zeros of Generalized Laguerre Polynomials	Weights	Weight times Exponential of Zero = $a^n_{kN} \exp(x^n_{kN})$
	$n = 5, N = 4$	
2.91078636936770048 (0)	2.96843923155522145 (1)	5.45339079699888520 (2)
6.00000000000000000 (0)	7.00000000000000000 (1)	2.82400155444914586 (4)
1.03341103467862578 (1)	1.97469504802402056 (1)	6.07501194316356380 (5)
1.67551032338460417 (1)	5.68657204207579951 (−1)	1.07522522732126438 (7)
	$n = 5, N = 8$	
1.78786047157082745 (0)	4.78286518603309507 (0)	2.85855186820191489 (1)
3.58823885764338781 (0)	3.37613267971511042 (1)	1.22115795065788180 (3)
5.92009002612269490 (0)	5.12313725300345871 (1)	1.90808809485486296 (4)
8.86684640118214593 (0)	2.54081265226598562 (1)	1.80216730854013967 (5)
1.25435889444281568 (1)	4.53758898116486575 (0)	1.27185211458041413 (6)
1.71405288833306908 (1)	2.74338324361631772 (−1)	7.62647046314642504 (6)
2.30233426297284257 (1)	4.37283100034351893 (−3)	4.36187637191007579 (7)
3.11295037859936706 (1)	8.82759451633260886 (−6)	2.91886557731016559 (8)
	$n = 5, N = 12$	
1.29748784000685508 (0)	1.13334726805127857 (0)	4.14815342846979275 (0)
2.58620943634406719 (0)	1.27102752975832523 (1)	1.68784065874957923 (2)
4.22697995386950259 (0)	3.59951248234067119 (1)	2.46602657834341949 (3)
6.25087547037452539 (0)	4.09672458660868638 (1)	2.12401457309494012 (4)
8.69028125595059872 (0)	2.21781320782776474 (1)	1.31845757933745043 (5)
1.15872437894981120 (1)	6.09556014277879247 (0)	6.56583037728656770 (5)
1.49995922575396278 (1)	8.58547769806400297 (−1)	2.80546343478958589 (6)
1.90103582323823253 (1)	5.98481941810913418 (−2)	1.07930634493047940 (7)
2.37456904925278601 (1)	1.89538249456284785 (−3)	3.89331206067960144 (7)
2.94144761288863084 (1)	2.30998625486114025 (−5)	1.37452950475636027 (8)
3.64163482053466644 (1)	7.74406887601347796 (−8)	5.06277099810570424 (8)
4.57744569372735530 (1)	3.01616108514055821 (−11)	2.28585454857795443 (9)
	$n = 5, N = 16$	
1.01975621006729237 (0)	3.51651938860095021 (−1)	9.74961600786600682 (−1)
2.02686733891905119 (0)	5.12310712750064588 (0)	3.88857731425075226 (1)
3.30047946506096635 (0)	2.04008497899245158 (1)	5.53386139750131562 (2)
4.85757645713809794 (0)	3.57964501907962966 (1)	4.60742858155172212 (3)
6.71275559761595590 (0)	3.32890537160876807 (1)	2.73913958593091106 (4)
8.88305877614446853 (0)	1.79120868308842187 (1)	1.29124780054447953 (5)
1.13897106617410895 (1)	5.81751150687781879 (0)	5.14310915694565138 (5)
1.42597409258898484 (1)	1.15803031442409197 (0)	1.80570468896330916 (6)
1.75281787111426743 (1)	1.40641304118632739 (−1)	5.76108359319635732 (6)
2.12414122811650920 (1)	1.01865573672515247 (−2)	1.71023416056506905 (7)
2.54627162726269377 (1)	4.21355482124525534 (−4)	4.81909868948838928 (7)
3.02820392701258409 (1)	9.27055945773008095 (−6)	1.31349522839105173 (8)
3.58350465381012185 (1)	9.67230928638172189 (−8)	3.53584440429819067 (8)
4.23454277633176611 (1)	3.93647895812592163 (−10)	9.67149353332157447 (8)
5.02404476674638389 (1)	4.29130683419748559 (−13)	2.82968933922063744 (9)
6.06147860634799663 (1)	4.84789862852962661 (−17)	1.02381299844288556 (10)

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Numerical Integration Over the Triangle

By Gerald E. Bartholomew

1. Introduction. Integration formulae for numerical integration over the simplex in n-space have been given by Hammer, Marlowe, and Stroud [1]. This paper describes a method for the evaluation of integrals over a triangle, based on a formula developed in [1]. The mid-points of the sides of the triangle are used as the points of evaluation in this method.

Other possibilities may be used; for example, the trisection points of the medians that are not the centroid could be used as the points of evaluation. These points are symmetrically spaced and the weights, w_i, again are equal. The centroid could also be used as a single point of evaluation with w_i equal to the area of the triangle. As another modification to this method, one could calculate the appropriate mid-points of the reference triangle for a reasonable number of subdivisions, then retain them as constants rather than calculate each point as it is used. Recently, (1958) Mr. Walter Leffin carried out computations by the use of these methods.

2. Mathematical Formulation. Consider the formula

$$\iint_T f(u, v)\, du\, dv \doteq \sum_i w_i f(u_i, v_i) \tag{1}$$

where T is a triangle, $w_i = A/3$ for all i, A being the area of T, and the points (u_i, v_i) are the mid-points of the sides of T. Formula (1) then gives a first approximation to the integral. Now divide T into four triangles by connecting the mid-points of the sides. Evaluate (1) over each subtriangle T' and sum. This gives a second approximation of the original integral. Iterate this process until the relation

$$| I_n - I_{n-1} | < \varepsilon, \tag{2}$$

is satisfied for some prescribed ε, where I_n is the nth approximation to the integral.

To find the desired mid-points of the sides of each triangle for successive subdivision, it was found convenient to subdivide a reference triangle R. Then the affine transformation

$$\begin{aligned} u &= a_{11}x + a_{12}y + a_{13} \\ v &= a_{21}x + a_{22}y + a_{23} \end{aligned} \tag{3}$$

will give the proper points of T for the evaluation of (1). R is defined by the vertices $(0, 0)$, $(1, -1)$, $(1, 1)$. The transformation (3) must then map $R \rightarrow T$ in the following manner

$$\begin{aligned} (0, 0) &\rightarrow (u_1, v_1) \\ (1, -1) &\rightarrow (u_2, v_2) \\ (1, 1) &\rightarrow (u_3, v_3) \end{aligned} \tag{4}$$

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where (u_1, v_1), (u_2, v_2), (u_3, v_3) are the vertices of T. Substituting (4) into (3) and solving for the A_{ij}, we obtain

$$a_{11} = \frac{u_2 + u_3 - 2u_1}{2} \qquad a_{21} = \frac{v_2 + v_3 - 2v_1}{2} \qquad a_{12} = \frac{u_3 - u_2}{2}$$

$$a_{22} = \frac{v_3 - v_2}{2} \qquad a_{13} = u_1 \qquad a_{23} = v_1. \tag{5}$$

Since the area of R is one unit, the absolute value of the Jacobian of (3) is the area of T. Note, also, that the area of each T' is one-fourth of the area of T. Hence the w_i are easily determined for each approximation.

The mid-points of the triangles in the previous $(n - 1)$ subdivisions together with their vertices constitute the vertices of the triangles of the nth subdivision. Hence one could find the mid-points for the nth subdivision by use of the formulae of analytic geometry. This is undesirable because of the large amount of information that must be retained. For example, the 5th approximation has 768 mid-points, 408 of them are distinct. Therefore, a procedure is used that greatly reduces the amount of information that must be retained after each subdivision. This method uses three incremental values, Δ_1, Δ_2, Δ_3, and the previously calculated mid-point, along with certain features of the triangle R to determine the next mid-point.

Since R is symmetrical about the x-axis, by reflection across the x-axis one generates a point $p_i^* = (x_i, -y_i)$ for every point $p_i = (x_i, y_i)$, where $y \neq 0$. Also every mid-point not lying on the boundary of R is a mid-point of two triangles. Hence one must check whether $(a)x = |y|$ or $(b)x = 1$ to determine whether it is an interior point with respect to R. The increments Δ_1, Δ_2, Δ_3 have

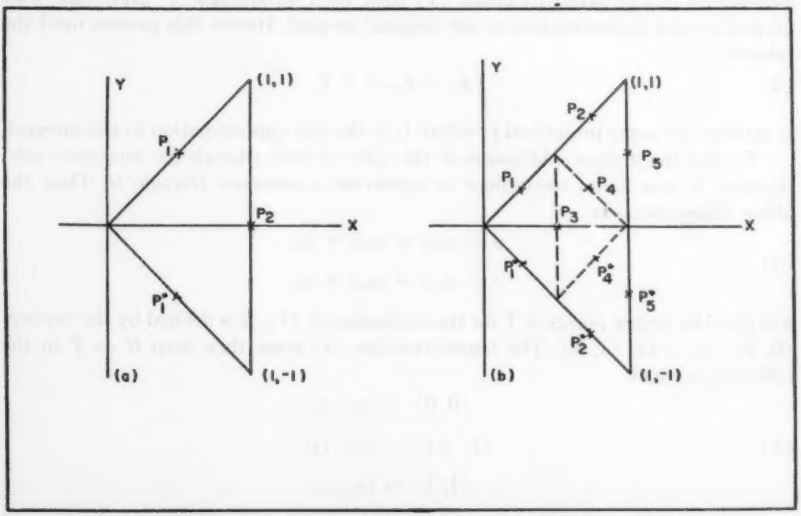

FIG. 1.—The sequence of point generation for first and second subdivisions.

the initial values $\frac{1}{2}$, 1, 0 respectively. Δ_1 and Δ_2 are divided by a factor, two, for each successive subdivision, while Δ_3 is altered during each subdivision and is reset to zero at the start of the next subdivision.

The flow diagram in section 3 will show specifically how the increments are used to generate the p_i. Figure 1 shows the order in which the points are generated for the first and second subdivisions. The points p_i are actually calculated along straight lines originating at the x-axis with slope one. Hence the obvious check must be made to see whether the next point should be calculated along the same line, or whether the next point should initiate the next line.

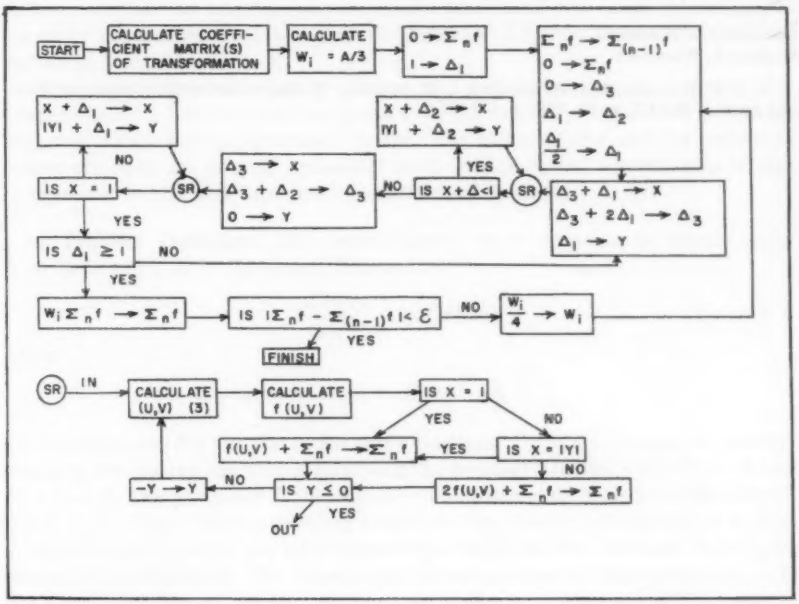

FIG. 2.—Flow diagram.

TABLE 1

f	Triangle	I	n	I_n	ε
$x^3 + y^3$	(0,0), (2,0), (2,1)	3.3000	6	3.3000009	10^{-5}
$\frac{1}{2 + y}$	R	.52323	2	.52316017	10^{-3}
$\sin x$	R	.60234	3	.60232905	10^{-3}
$\frac{1}{\sqrt{x}}$	R	1.3333	4	1.3307964	10^{-2}
$f = \begin{cases} 1 \text{ for } x - y < \frac{1}{2} \\ 2 \text{ for } x - y \geq \frac{1}{2} \end{cases}$	R	1.5625	4	1.5937500	10^{-1}

3. Flow diagram. Figure 2 contains the flow diagram. The notation used in the flow diagram should be explained. The expression $a \rightarrow b$ means that a replaces b. For example, $1 \rightarrow \Delta_1$ means that Δ_1 is set equal to one. $\sum_n f$ is the partial calculation of I_n that has already been made, or it is I_n if the calculation is completed for the nth approximation. $\sum_{(n-1)} f$ is then I_{n-1}.

4. Results of Trial Calculations. A code was devised for the IBM 650 and several trial calculations were run. Table 1 gives the results that were obtained. It might be pointed out that formula (1) is exact in the first approximation for a quadratic function. This problem was suggested by Professor Preston C. Hammer.

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Efficient Method for Solving Atomic Schroedinger's Equation

By Sherwood Skillman

1. Introduction. One of the basic numerical problems in atomic quantum theory is the solution of the Schroedinger's wave equation for a spherically symmetric potential. In practice, one is usually concerned with such potentials obtained by the Hartree-Fock self-consistent fields [1] or by the Thomas-Fermi-Dirac statistical field methods [2, 3].

This paper describes a highly efficient and rapidly convergent technique for solving the radial Schroedinger's equation for an arbitrary atomic-like potential. The method has been programmed for the IBM 650 computer and the numerical results obtained are in good agreement both with pertinent experimental results (x-ray term levels) and with previous theoretical work [4].

2. Problem Definition. The Schroedinger's wave equation in atomic units (r in Bohr radii, E in Rydbergs) is given by

$$\frac{d^2P(r)}{dr^2} = q(r)P(r) \qquad \text{for} \quad r \geqq 0$$

where

$$q(r) = V(r) - E + \frac{l(l+1)}{r^2}.$$

In this equation $P(r)$ is the radial wave function, $V(r)$ is the potential energy, and E is the desired eigenvalue satisfying the boundary conditions $P(0) = 0$ and $P(\infty) = 0$. The parameter l, the angular-momentum number, is an integral number 0, 1, 2, 3, etc. These numerical values of l are usually designated as s, p, f, d, etc. For each l, there is a set of eigenvalues satisfying the boundary values and labeled by an integer n. The lowest state for each l value is taken as $n = l + 1$, the principal quantum number, and for every value of n there is a distinct energy value E. The various radial wave functions and associated eigenvalues are specified using the subscript nl.

3. Integration Method. The integration mesh was made flexible for many different type atoms by choosing it as a function of the atomic structure. The integrating variable x was made equal to r/μ, where

$$\mu = \frac{3\pi}{4(6\pi z)^{1/3}}$$

and z = the atomic number. It was found that $\Delta x = .0025$ was a satisfactory starting step and the mesh could be doubled every 40 points. The potential values were calculated for the same mesh using the Fermi-Thomas-Dirac equation and the desired boundary conditions.

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FLOW CHART FOR SOLVING SCHROEDINGER'S EQUATION

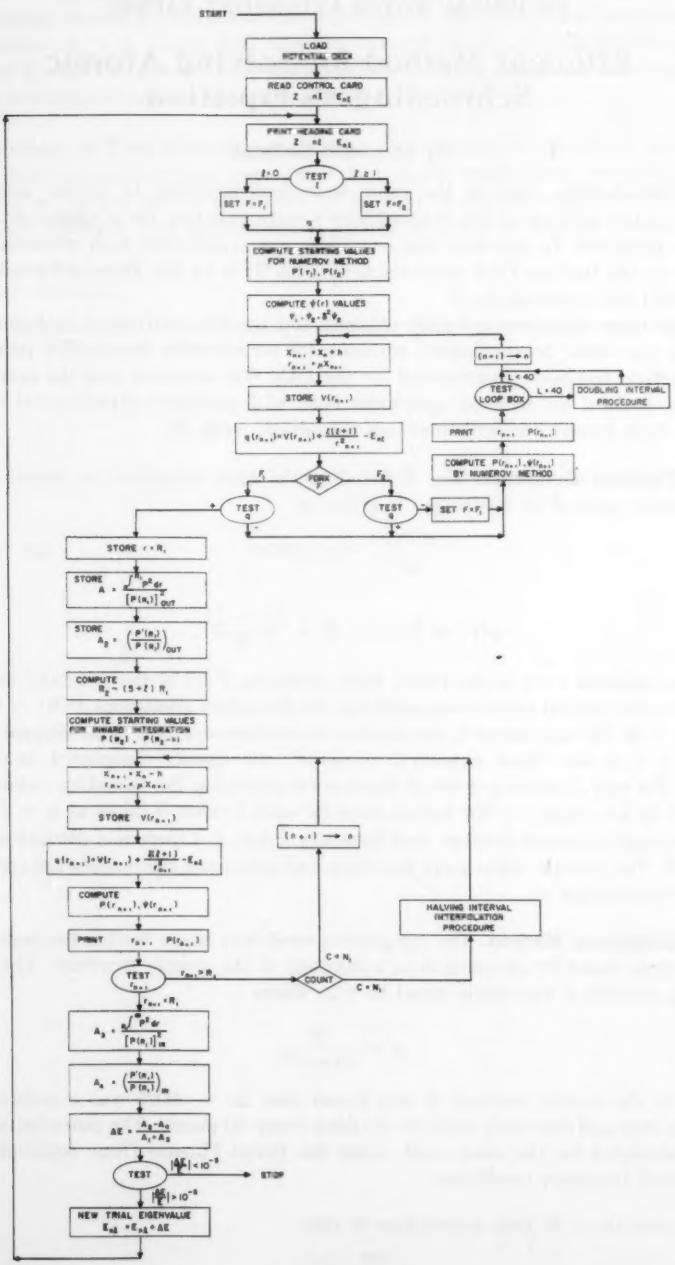
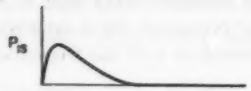


FIG. 1

CONVERGENCE OF TRIAL EIGENVALUES

ATOM: Germanium
QUANTUM STATE: 1s

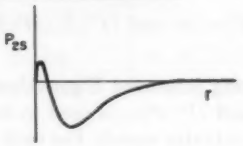


TRIAL	E_{1s}	ΔE_{1s}
1	− 700.000	− 86.824
2	− 786.824	− 14.834
3	− 801.658	− .230
4	− 801.888	+ 000 13

FIG. 2

CONVERGENCE OF TRIAL EIGENVALUES

ATOM: Germanium
QUANTUM STATE: 2s

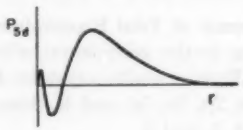


TRIAL	E_{2s}	ΔE_{2s}
1	− 150.000	+ 57.042
2	− 92.958	− 7.956
3	− 100.914	− 1.150
4	− 102.064	− .015
5	− 102.079	− .000 02

FIG. 3

CONVERGENCE OF TRIAL EIGENVALUES

ATOM: Cerium
QUANTUM STATE: 5d



TRIAL	E_{5d}	ΔE_{5d}
1	− 2.22 000	+ .28 567
2	− 1.93 433	− .03 142
3	− 1.96 575	− .00 126
4	− 1.96 701	− .00 000 3

FIG. 4

A method applicable only to a linear second-order differential equation which lacks the first derivative was used to solve Schroedinger's equation. This method introduced by Numerov [5] is approximately four times faster than the Runge-Kutta-Gill procedure and the truncation error is proportional to h^6.

4. Matching Procedure. Schroedinger's equation is integrated outward to a point where $P(r)$ changes from a traveling-wave behavior to a damped-wave behavior. The choice of this matching radius R_1 is made dependent upon the value of $q(r)$, the kinetic energy, becoming negative. This choice makes the program quite general as it will handle any value of l and E. The integral $\int_0^{R_1} P^2(r)\,dr$ and the logarithmic derivative of $P(r)$ at $r = R_1$, $(P'(r)/P(r))_{\text{out}}$ are stored.

Then an inward integration is performed starting at $R_2 = (5 + l)R_1$. At R_2 the final normalized wave function is essentially zero, thus satisfying the second boundary condition $P(\infty) = 0$. Since q in the neighborhood of R_2 is essentially a constant, a good approximation for $P(R_2) = e^{-R_2\sqrt{q(R_2)}}$. This value is then used to start the inward integration, which is continued until R_1 is reached. The values of $\int_{R_1}^{\infty} P^2(r)\,dr$ and $(P'(R_1)/P(R_1))_{\text{in}}$ are calculated and stored.

5. Calculating Improved Eigenvalue. The difference at R_1 between the values of $(P'/P)_{\text{out}}$ and $(P'/P)_{\text{in}}$ is used to estimate what change in E, (ΔE), is necessary to obtain a better match. For each trial eigenvalue an outward and an inward integration is performed. A formula developed by Hartree (1) is then used to obtain ΔE as follows:

$$\left\{\frac{\int_0^{R_1} P^2(r)\,dr}{P^2(R_1)_{\text{out}}} + \frac{\int_{R_1}^{\infty} P^2(r)\,dr}{P^2(R_1)_{\text{in}}}\right\}\Delta E = \left[\left(\frac{P'}{P}\right)_{\text{out}} + \left(\frac{P'}{P}\right)_{\text{in}}\right]$$

The value ΔE is then added to the trial E and the process repeated until $|\Delta E/E| < 10^{-5}$. The radial wave functions are then made continuous at R_1 and normalized.

6. Flow-Chart of Procedure. The main features of the program are outlined in Fig. 1.

7. Convergence of Trial Eigenvalues. The program has been used in applications pertaining to the germanium, silicon, and cerium atoms and also the zinc-sulfide and gallium-arsenide crystals. It has been used for the seven quantum states 1s, 2s, 3s, 2p, 3p, 5d, and 4f. Some typical eigenvalues convergence rates are shown in Fig. 2, 3, and 4.

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Explicit Formulae for 25 of the Associated Legendre Functions of the Second Kind

By Dietrich Suschowk

Since the present writer knows of no place where to find explicit expressions for some more of Legendres associated functions of the second kind than are given for instance in [1] or [2], he hopes the small collection of formulae at the end of this note might be of some use to others.

The associated Legendre functions $Q_n^k(x)$ of the second kind for $|x| > 1$ and nonnegative integer values of n, k can be defined by [1]

$$Q_n^k(x) = \frac{(x^2-1)^{k/2}}{2^n n!}\frac{d^{n+k}}{dx^{n+k}}\left[(x^2-1)^n \log\frac{x+1}{x-1}\right] - \frac{1}{2}\frac{d^k}{dx^k}\left[\log\frac{x+1}{x-1}\cdot\frac{d^n}{dx^n}(x^2-1)^n\right].$$

These functions can be written as follows:

$$(1)\qquad Q_n^k(x) = \begin{cases} \dfrac{(x^2-1)^{-k/2}}{(n-k)!}\left[\frac{1}{2}A_{nk}^{(1)}(x)\log\dfrac{x+1}{x-1} + A_{nk}^{(2)}(x)\right] & (n \geqq k) \\ \dfrac{(-1)^k}{(x^2-1)^{k/2}}A_{nk}^{(2)}(x) & (n < k) \end{cases}$$

where $A_{nk}^{(i)}(x)$ $(i = 1, 2)$ is a polynomial and

$$\text{degree of } A_{nk}^{(1)}(x) = \begin{cases} n+k & \text{for } n \geqq k \\ \text{no degree for } n < k\ (A_{nk}^{(1)}(x) \equiv 0) \end{cases}$$

$$\text{degree of } A_{nk}^{(2)}(x) = \begin{cases} n+k-1 & \text{for } n \geqq k \\ k-n-1 & \text{for } n < k \end{cases}$$

(2) If the degree of $A_{nk}^{(i)}(x)$ is even (uneven), $A_{nk}^{(i)}(x)$ contains only even (uneven) powers of x.

(3) The coefficients of $A_{nk}^{(i)}(x)$ are integers.

(4) For each fixed triple (n, k, i) and $n \geqq k$ the coefficients of $A_{nk}^{(i)}(x)$ have alternating signs and the coefficient of the highest power in $A_{nk}^{(1)}(x)$ is always positive and in $A_{nk}^{(2)}(x)$ always negative. For $n < k$ the coefficients of $A_{nk}^{(2)}(x)$ are all positive.

(5) If $n \geqq k > 0$, $A_{nk}^{(1)}(x)$ has a zero of nth order for $x = 1$.

The formulae are as follows: Let $\tau = \frac{1}{2}\log(x+1)/(x-1)$. Then

$$Q_0^0(x) = \tau$$

$$Q_1^0(x) = x\tau - 1$$

$$Q_2^0(x) = \tfrac{1}{2}[(3x^2-1)\tau - 3x]$$

$$Q_3^0(x) = \frac{1}{3!}[(15x^3-9x)\tau - 15x^2 + 4]$$

$$Q_4^0(x) = \frac{1}{4!}[(105x^4-90x^2+9)\tau - 105x^3 + 55x]$$

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$$Q_0^1(x) = -\frac{1}{\sqrt{x^2-1}}$$

$$Q_1^1(x) = \frac{1}{\sqrt{x^2-1}}\,[(x^2-1)\tau - x]$$

$$Q_2^1(x) = \frac{1}{\sqrt{x^2-1}}\,[(3x^3-3x)\tau - 3x^2 + 2]$$

$$Q_3^1(x) = \frac{1}{2\sqrt{x^2-1}}\,[(15x^4-18x^2+3)\tau - 15x^3 + 13x]$$

$$Q_4^1(x) = \frac{1}{3!\sqrt{x^2-1}}\,[(105x^5-150x^3+45x)\tau - 105x^4 + 115x^2 - 16]$$

$$Q_0^2(x) = \frac{2x}{x^2-1}$$

$$Q_1^2(x) = \frac{2}{x^2-1}$$

$$Q_2^2(x) = \frac{1}{x^2-1}\,[(3x^4-6x^2+3)\tau - 3x^3 + 5x]$$

$$Q_3^2(x) = \frac{1}{x^2-1}\,[(15x^5-30x^3+15x)\tau - 15x^4 + 25x^2 - 8]$$

$$Q_4^2(x) = \frac{1}{2(x^2-1)}\,[(105x^6-225x^4+135x^2-15)\tau - 105x^5 + 190x^3 - 81x]$$

$$Q_0^3(x) = -\frac{6x^2+2}{(x^2-1)^{3/2}}$$

$$Q_1^3(x) = -\frac{8x}{(x^2-1)^{3/2}}$$

$$Q_2^3(x) = -\frac{8}{(x^2-1)^{3/2}}$$

$$Q_3^3(x) = \frac{1}{(x^2-1)^{3/2}}\,[(15x^6-45x^4+45x^2-15)\tau - 15x^5 + 40x^3 - 33x]$$

$$Q_4^3(x) = \frac{1}{(x^2-1)^{3/2}}\,[(105x^7-315x^5+315x^3-105x)\tau$$

$$- 105x^6 + 280x^4 - 231x^2 + 48]$$

$$Q_0^4(x) = \frac{24x^3 + 24x}{(x^2 - 1)^2}$$

$$Q_1^4(x) = \frac{40x^2 + 8}{(x^2 - 1)^2}$$

$$Q_2^4(x) = \frac{48x}{(x^2 - 1)^2}$$

$$Q_3^4(x) = \frac{48}{(x^2 - 1)^2}$$

$$Q_4^4(x) = \frac{1}{(x^2 - 1)^2}[(105x^8 - 420x^6 + 630x^4 - 420x^2 + 105)\tau - 105x^7 + 430x^5 - 511x^3 + 231x].$$

The above formulae have been calculated by use of the recurrence relations [2]

$$Q_n^{k+2}(x) = (n - k)(n + k - 1)Q_n^k(x) - 2(k + 1)\frac{x}{\sqrt{x^2 - 1}}Q_n^{k+1}(x)$$

$$(n - k + 1)Q_{n+1}^k(x) = (2n + 1)xQ_n^k(x) - (n + k)Q_{n-1}^k(x)$$

$$\sqrt{x^2 - 1}\,Q_n^{k+1}(x) = (n - k)xQ_n^k(x) - (n + k)Q_{n-1}^k(x)$$

starting with

$$Q_0^0(x) = \tau, \qquad Q_0^1(x) = -\frac{1}{\sqrt{x^2 - 1}}, \qquad Q_1^0(x) = x\tau - 1.$$

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A Short Table of $\int_x^\infty J_0(t)t^{-n}\,dt$ and $\int_x^\infty J_1(t)t^{-n}\,dt$

By I. M. Longman

1. Introduction. In various physical applications, and in particular in geophysics, there arises a need for the numerical evaluation of integrals of the type

$$L = \int_x^\infty J_0(t)g(t)\,dt \tag{1}$$

and

$$M = \int_x^\infty J_1(t)g(t)\,dt \tag{2}$$

where x is a positive number, and where $g(t)$ is, at least for sufficiently large t, a monotonically decreasing function of t. Here $J_0(t)$, $J_1(t)$ denote the Bessel functions (of the first kind) of orders zero and one, respectively.

A method of computation of (1) and (2) has been given by the author [1] for the case $x = 0$, and this method can also be applied to the evaluation of (1) and (2) for small values of x (such that the interval $0 - x$ is within the first few cycles of J_0 and J_1 as the case may be), even if the integrals diverge for $x = 0$. The present paper presents an alternate method (which is easily extended to large values of x) for the case where $g(t)$ is a function of the type that can be expanded as a series of inverse powers of t

$$g(t) = a_0 + a_1t^{-1} + a_2t^{-2} + \cdots, \tag{3}$$

when t is sufficiently large.

2. Description of the Method. Suppose, then, that we wish to evaluate an integral of the type (1) or (2), where for simplicity we will suppose that an expansion of the form (3) is valid when $t > x$. Then our integrals can be expanded in the forms

$$L = a_0\int_x^\infty J_0(t)\,dt + a_1\int_x^\infty J_0(t)t^{-1}\,dt + a_2\int_x^\infty J_0(t)t^{-2}\,dt + \cdots, \tag{4}$$

$$M = a_0\int_x^\infty J_1(t)\,dt + a_1\int_x^\infty J_1(t)t^{-1}\,dt + a_2\int_x^\infty J_1(t)t^{-2}\,dt + \cdots, \tag{5}$$

so that (1), (2) can be evaluated if we can evaluate integrals of the type

$$L_n = \int_x^\infty J_0(t)t^{-n}\,dt, \qquad n = 0, 1, 2, \cdots \tag{6}$$

and

$$M_n = \int_x^\infty J_1(t)t^{-n}\,dt, \qquad n = 0, 1, 2, \cdots. \tag{7}$$

We will see later that the series (4), (5) are quite rapidly convergent for $x > 1$, the rapidity of convergence increasing with x. However it should be noted that in

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a number of applications (for example propagation in layered media, seismology problems) the integrals we need to evaluate have ρt in place of t in the arguments of the Bessel functions, where ρ is a positive number which may be as high as 20. After making the appropriate scale-factor transformation we obtain the integrals (6) and (7) with a factor ρ^{n-1} in front of each integral. For this reason the integrals (6) and (7) are often required with high precision and at least ten place accuracy would be needed for such application.

With regard to the integrals L_n, M_n, the only published tables known to the author are one of

$$\int_0^x J_0(t)\,dt, \qquad x = 0(0.01)10$$

by Lowan and Abramowitz [2], from which we can obtain L_0 as

$$L_0 = 1 - \int_0^x J_0(t)\,dt,$$

and one of L_1, for $x = 0\ (0.1)\ 10\ (1)\ 22$, by Lowan, Blanch and Abramowitz, [3]. Watson [4] gives a table of the maxima and minima of

$$\int_0^x J_0(t)\,dt$$

and this table has been extended by the author [1], who has also pointed out an error in Watson's table. Also Smith [5] and Lowan, Blanch and Abramowitz [3] give asymptotic expressions for L_1. For large values of x beyond the range of the published tables we can use the series

$$\int_0^x J_0(t)\,dt = 2[J_1(x) + J_3(x) + J_5(x) + \cdots]. \tag{8}$$

Also we have the obvious result

$$M_0 = J_0(x). \tag{9}$$

It is not the purpose of this paper to give extensive tables, but rather to present a general method for their computation, and to illustrate their use by way of an example.

3. Method of Computation. Integration by parts show that the L_n, M_n satisfy the following recurrence relations for a given value of x:

$$L_n = -x^{-n}J_1(x) + (n+1)M_{n+1} \tag{10}$$

$$M_n = x^{-n}J_0(x) - nL_{n+1} \tag{11}$$

from which we deduce

$$L_{n+2} = \frac{J_0(x)}{(n+1)x^{n+1}} - \frac{J_1(x)}{(n+1)^2x^n} - \frac{L_n}{(n+1)^2} \tag{12}$$

and

$$M_{n+2} = \frac{J_0(x)}{n(n+2)x^n} + \frac{J_1(x)}{(n+2)x^{n+1}} - \frac{M_n}{n(n+2)}. \tag{13}$$

Equation (12) has been given by Smith [5] where he uses it to obtain an asymptotic expansion for L_1. Thus starting from L_0 and L_1, which are either obtained

from existing tables, or must be calculated, we can use (12) to compute successive L_n's. Furthermore the form (12) shows that for even moderately large x, L_n tends to zero very rapidly as n increases. This makes the calculation of the series (4) a very rapid process.

With regard to the M_n's, starting with the relations

$$M_0 = J_0(x)$$
$$M_1 = L_0 + J_1(x)$$
$$M_2 = \tfrac{1}{2}(L_1 + x^{-1}J_1(x))$$

(which are obtainable from (10), (11)), successive M_n's can be computed by means of (13). Some L_n's and M_n's have been computed in this way, and these are given in tables 1 and 2.

With regard to L_0 and L_1, these are obtainable (by interpolation if necessary) from existing tables ([1], [2], [3], [4], [5]), or, for large x, by means of asymptotic expansions. The interpolation can easily be carried out accurately using Taylor's theorem. Examples of such interpolation are given in Longman [1]. Asymptotic series for L_1 are given by Smith [5] and by Lowan, Blanch and Abramowitz [3], while for L_0 we can apply Smith's method to obtain the result

$$\int_x^\infty J_0(t)\,dt \sim J_0(x)\left[\frac{1}{x} - \frac{1^2.3}{x^3} + \frac{1^2.3^2.5}{x^5} - \frac{1^2.3^2.5^2.7}{x^7} + \cdots\right] - J_1(x)\left[1 - \frac{1^2}{x^2} + \frac{1^2.3^2}{x^4} - \frac{1^2.3^2.5^2}{x^6} + \cdots\right], \tag{14}$$

and for large x this can be used to compute L_0 using the known asymptotic series for $J_0(X)$, $J_1(X)$.

4. Tables. Tables 1 and 2 were computed from equations (10), (11), (12) and (13) and existing tables [2] and [3] of L_0 and L_1 by the use of an IBM 709 computer at the Western Data Processing Center of the University of California, Los Angeles.

5. Example of the Use of the Tables. Suppose we wish to compute

$$I = \int_{10}^\infty [J_0(t)/(t^2 + 1)]\,dt.$$

Using the expansion

$$(t^2 + 1)^{-1} = t^{-2} - t^{-4} + t^{-6} - \cdots, \qquad (t > 1)$$

we have

$$I = L_2 - L_4 + L_6 - L_8 + \cdots,$$

from which we obtain

$$I = -0.001042.$$

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TABLE 1. $L_n = \int_x^\infty J_0(t)t^{-n}\,dt$

	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$	$x=7$	$x=8$	$x=9$	$x=10$
L_0	0.080270	−0.425770	−0.387567	−0.024734	0.284688	0.293779	0.045360	−0.210747	−0.252266	−0.067011
L_1	0.237097	−0.135296	−0.126534	−0.023126	0.046841	0.049422	0.011431	−0.022961	−0.028159	−0.008787
L_2	0.244878	−0.039009	−0.038176	−0.008510	0.007372	0.008013	0.002192	−0.002433	−0.003082	−0.001055
L_3	0.213312	−0.010280	−0.011069	−0.002502	0.001117	0.001265	0.000372	−0.000251	−0.000332	−0.000120
L_4	0.178963	−0.002357	−0.003155	−0.000664	0.000163	0.000196	0.000059	−0.000025	−0.000035	−0.000013
L_5	0.150464	−0.000365	−0.000896	−0.000167	0.000023	0.000030	0.000009	−0.000002	−0.000004	−0.000001
L_6	0.128279	+0.000052	−0.000255	−0.000041	0.000003	0.000005	0.000001	−0.000000	−0.000000	−0.000000
L_7	0.111130	0.000093	−0.000073	−0.000010	0.000000	0.000001	0.000000			
L_8	0.097715	0.000065	−0.000021	−0.000002		0.000000				
L_9	0.087037	0.000037	−0.000006	−0.000001						
L_{10}	0.078383	0.000020	−0.000002	−0.000000						
L_{11}	0.071249	0.000010	−0.000001							
L_{12}	0.065279	0.000005	−0.000000							
L_{13}	0.060216	0.000002								
L_{14}	0.055871	0.000001								
L_{15}	0.052105	0.000001								
L_{16}	0.048809	0.000000								
L_{17}	0.045902									
L_{18}	0.043320									
L_{19}	0.041011									
L_{20}	0.038935									
L_{21}	0.037057									
L_{22}	0.035352									

Blank spaces denote zero entries.

TABLE 2. $M_n = \int_x^\infty J_1(t)t^{-n}\,dt$

	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$	$x=7$	$x=8$	$x=9$	$x=10$
M_0	0.765198	0.223891	−0.260052	−0.397150	−0.177597	0.150645	0.300079	0.171651	−0.090334	−0.245936
M_1	0.520320	0.150955	−0.048508	−0.090778	−0.042891	0.017095	0.040677	0.023890	−0.006955	−0.023539
M_2	0.338574	0.076533	−0.006757	−0.019818	−0.009338	0.001654	0.005381	0.003184	−0.000451	−0.002220
M_3	0.228309	0.035057	−0.000167	−0.004213	−0.001910	0.000109	0.000699	0.000411	−0.000018	−0.000207
M_4	0.163341	0.015453	+0.000372	−0.000883	−0.000376	−0.000004	0.000089	0.000052	+0.000001	−0.000019
M_5	0.123803	0.006738	0.000206	−0.000184	−0.000072	−0.000003	0.000011	0.000006	0.000000	−0.000002
M_6	0.098419	0.002943	0.000083	−0.000039	−0.000014	−0.000001	0.000001	0.000001		
M_7	0.081190	0.001295	0.000030	−0.000008	−0.000003	−0.000000	0.000000	0.000000		
M_8	0.068898	0.000575	0.000010	−0.000002	−0.000000					
M_9	0.059752	0.000258	0.000003	−0.000000						
M_{10}	0.052709	0.000116	0.000001							
M_{11}	0.047130	0.000053	0.000000							
M_{12}	0.042608	0.000024								
M_{13}	0.038872	0.000011								
M_{14}	0.035733	0.000005								
M_{15}	0.033061	0.000002								
M_{16}	0.030760	0.000001								
M_{17}	0.028756	0.000001								
M_{18}	0.026997	0.000000								
M_{19}	0.025441									
M_{20}	0.024053									
M_{21}	0.022809									
M_{22}	0.021687									

Blank spaces denote zero entries

1. I. M. LONGMAN, "Tables for the rapid and accurate numerical evaluation of certain infinite integrals involving Bessel functions," *MTAC*, v. 11, 1957, p. 166.

2. ARNOLD N. LOWAN & MILTON ABRAMOWITZ, "Table of the integrals $\int_0^x J_0(t)\,dt$ and $\int_0^x Y_0(t)\,dt$, *Tables of Functions and of Zeros of Functions*, NBS Applied Mathematics Series, No. 37, U. S. Government Printing Office, Washington, D. C., 1954, p. 21.

3. ARNOLD N. LOWAN, G. BLANCH, & MILTON ABRAMOWITZ, "Table of $Ji_0(x) = \int_x^\infty J_0(t)/t\,dt$ and related functions," *Ibid.*, p. 33.

4. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, The University Press, Cambridge, 1948, p. 752.

5. V. G. SMITH, "An asymptotic expansion of $Ji_0(x) = \int_x^\infty J_0(t)/t\,dt$," *Journal of Mathematics and Physics*, v. 22, 1943, p. 58.

Numerical Integration of $\int_0^\infty e^{-x} J_0\left(\frac{\eta x}{\xi}\right) J_1\left(\frac{x}{\xi}\right) x^{-n}\, dx$

By Gerard P. Weeg

1. Introduction. This integral arises in a certain thermo-elastic study, and although it is similar to many evaluated in Watson's "Bessel Functions," this integral is not listed. Numerical integration techniques were applied. By means of Simpson's rule with the limits of integration set at 0 to 12 and making use of 400 points, the above integral was evaluated to four decimal places for seven different combinations of η and ξ.

The same integral was evaluated by the Gauss-Laguerre formula. Although the integrals in question can be expressed in terms of complicated elliptic integrals, the natural method of evaluation, to avoid lengthy analysis, is by quadrature. It is deemed that the experience with the Gauss-Laguerre formula is worth reporting, even though the integrals are of a special nature. Using the Gauss-Laguerre formula with only six points, the results agreed with Simpson's rule entries to three decimal places, although the upper bound of the truncation error associated with the Gauss-Laguerre formula was greater than unity.

2. Numerical Results. The integral under consideration is

$$I(\eta, \xi, n) = \int_0^\infty e^{-x} J_0\left(\frac{\eta x}{\xi}\right) J_1\left(\frac{x}{\xi}\right) x^{-n}\, dx \tag{1}$$

where η and ξ are integers and n is 0 or 1. A careful analysis of Simpson's rule applied to $I(\eta, \xi, 0)$ indicates accuracy to within a unit in the fifth decimal place in the values listed in Table 1. It is possible to show* that

$$I(\eta, \xi, 0) + I(\eta, \xi, 1) = \frac{1}{\pi\sqrt{\xi}} [Q_{-1/2}(x) - \eta Q_{1/2}(x)] \tag{2}$$

where $x = (1 + \xi^2 + \eta^2)/2\eta$ and $Q_m(x)$ is the Legendre function of the second kind. Use of (2) then verifies that the values of $I(\eta, \xi, 1)$ are also correct to the number of places given in Table 1.

TABLE 1

ξ	η	$I(\eta, \xi, 0)$	$I(\eta, \xi, 1)$	Maximum Error in $I(\eta, \xi, 0)$
1	1	.17868	.33880	1.12×10^{-5}
1	3	.01741	.15950	8.37×10^{-6}
1	6	.00229	.08246	7.97×10^{-6}
3	1	.13463	.15486	2.48×10^{-5}
3	3	.05948	.11742	8.37×10^{-6}
6	1	.07850	.08167	1.47×10^{-4}
6	6	.02954	.05887	7.97×10^{-6}

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* The author is indebted to Dr. Gertrude Blanch for this relationship.

$I(\eta, \xi, 0)$ was also evaluated for $\xi = \eta = 1$ by means of the Gauss-Laguerre formula. Using four points the results agree to two decimal places with those in Table 1. Using six points the results agree to three places. In both cases, the truncation error found exceeded unity.

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1. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, The Macmillan Company, New York, 1945.

Calculation of Gamma Functions to High Accuracy

By M. E. Sherry and S. Fulda*

The two constants, $\Gamma\left(\frac{1}{3}\right)$ and $\Gamma\left(\frac{2}{3}\right)$ have been calculated to 35 decimal places on the Cambridge Computer by means of an interpretive routine that treats floating-point numbers of 37 significant digits [1, 2].

Stirling's asymptotic expansion for $\ln \Gamma(x)$ can be written

$$\ln \Gamma(x) \sim (x - \tfrac{1}{2}) \ln x - x + \tfrac{1}{2} \ln 2\pi + \sum_{r=1}^{\infty} \frac{C_r}{x^{2r-1}}. \tag{1}$$

With the remainder of the series made very small, the accuracy of $\ln \Gamma(x)$ depends primarily on the accuracy of $\ln 2\pi$, $\ln x$, and the C_r. Uhler [3, 4] has published the $\ln x$ for all primes through 101, as well as the $\ln \pi$, to more than 100 significant figures. Uhler has also calculated the C_r to over 100 significant figures [5].

The recursion formula for the Gamma function

$$\Gamma(x + 1) = x\Gamma(x) \tag{2}$$

can be extended to

$$\Gamma(x + n) = (x)_n \Gamma(x) \tag{3}$$

where $(x)_n$ is the Pochhammer-Barnes symbol

$$(x)_n = (x + n - 1)(x + n - 2) \cdots (x + 1)x. \tag{4}$$

If we take the logarithm of Eq. (3),

$$\ln \Gamma(x + n) = \ln \Gamma(x) + \sum_{j=0}^{n-1} \ln (x + j). \tag{5}$$

Substituting Eq. (1) into Eq. (5) gives, after solving for $\ln \Gamma(x)$,

$$\ln \Gamma(x) \sim \lambda(x) + \sum_{r=1}^{\infty} \frac{C_r}{(x + n)^{2r-1}}, \tag{6}$$

where

$$\lambda(x) = (x + n - \tfrac{1}{2}) \ln (x + n) - (x + n) + \tfrac{1}{2} \ln 2\pi - \sum_{j=0}^{n-1} \ln (x + j). \tag{7}$$

The term $\lambda(x)$ is calculated quite easily by hand, and it is not difficult to calculate $\ln \Gamma(x)$ from Eq. (6) with a digital computer.

To calculate $\Gamma(x)$, let

$$\Gamma(x) = e^{\psi} e^{\ln \Gamma(x) - \psi} \tag{8}$$

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* S. Fulda, Bell Telephone Laboratories, Murray Hill, New Jersey, co-authored this paper while on temporary duty at the Air Force Cambridge Research Center.

where

$$\psi = \pm 0.1n \tag{9}$$

is the exponent of e nearest in value to $\ln \Gamma(x)$. Values of $e^{\pm 0.1n}$, n being any integer from 0 to 10, have been tabulated by Van Orstrand [6].

(10) Since $$|\ln \Gamma(x) - \psi| \leqq 0.05$$

it will always be possible to calculate $e^{\ln \Gamma(x)}$ by means of the exponential power series expansion using very few terms.

These techniques led to the following values:

$$\Gamma(\tfrac{1}{3}) = 2.6789\ 38534\ 70774\ 76336\ 55692\ 94097\ 46776$$

$$\Gamma(\tfrac{2}{3}) = 1.3541\ 17939\ 42640\ 04169\ 45288\ 02815\ 45138$$

$$\ln \Gamma(\tfrac{1}{3}) = .98542\ 06469\ 27767\ 06918\ 71740\ 36977\ 96139$$

$$\ln \Gamma(\tfrac{2}{3}) = .30315\ 02751\ 47523\ 56867\ 58628\ 17372\ 01104.$$

It was possible to examine round-off and truncation errors at each step in the calculation. The final relative error was less than $\pm 3 \times 10^{-35}$ in each case.

As a final independent check, the values of $\Gamma(\frac{1}{3})$ and $\Gamma(\frac{2}{3})$ were put into

$$\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3}) = \frac{2\pi}{\sqrt{3}}. \tag{11}$$

The error analysis was consistent with this identity.

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1. B. Zondek, "The values of $\Gamma(\frac{1}{3})$ and $\Gamma(\frac{2}{3})$ and their logarithms accurate to 28 decimals," *MTAC*, v. 9, 1955, p. 24–25.
2. Air Force Cambridge Research Center Report, *Cambridge Computer Interpretive Routine for Quadruple Precision Numbers*, Series 3, TN-59-155, 1959.
3. H. S. Uhler, "Natural logarithms of small primary numbers," *Proc.*, Nat. Acad. Sci., v. 29, 1943, p. 319–325.
4. H. S. Uhler, "Log π and other basic constants," *Proc.*, Nat. Acad. Sci., v. 24, 1938, p. 23–30.
5. H. S. Uhler, "The coefficients of Stirling's series for log $\Gamma(x)$," *Proc.*, Nat. Acad. Sci., v. 28, 1942, p. 59–62.
6. C. E. Van Orstrand, "Tables of the exponential function and of the circular sine and cosine to radian argument," *Mem.*, Nat. Acad. Sci., v. 14: 5, 1921 (Tables IV and V).

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

49 [F, Z].—MARTIN DAVIS, *Computability and Unsolvability*, McGraw-Hill Book Company, Inc., New York, 1948, xxv + 210 p., 23 cm. Price $7.50.

The following is a quotation from the author's preface.

"This book is an introduction to the theory of computability and noncomputability, usually referred to as the theory of recursive functions. This subject is concerned with the existence of purely mechanical procedures for solving problems. Although the theory is a branch of pure mathematics, it is, because of its relevance to certain philosophical questions and to the theory of digital computers, of potential interest to nonmathematicians. The existence of absolutely unsolvable problems and the Gödel incompleteness theorem are among the results in the theory of computability which have philosophical significance. The existence of universal Turing machines, another result of the theory, confirms the belief of those working with digital computers that it is possible to construct a single 'all-purpose' digital computer on which can be programmed (subject of course to limitations of time and memory capacity) any problem that could be programmed for any conceivable deterministic digital computer. This assertion is sometimes heard in the strengthened form: anything that can be made completely precise can be programmed for an all-purpose digital computer. However, in this form, the assertion is false. In fact, one of the basic results of the theory of computability (namely, the existence of nonrecursively enumerable sets) may be interpreted as asserting the possibility of programming a given computer in such a way that it is impossible to program a computer (either a copy of the given computer or another machine) so as to determine whether or not a given item will be a part of the output of the given computer. Another result (the unsolvability of the halting problem) may be interpreted as implying the impossibility of constructing a program for determining whether or not an arbitrary given program is free of 'loops.'

"Because it was my aim to make the theory of computability accessible to persons of diverse backgrounds and interests, I have been careful (particularly in the first seven chapters) to assume no special mathematical training on the reader's part."

The author succeeds admirably in presenting a readable, motivated, yet not verbose, exposition of the theory of recursive functions. The not mathematically mature reader will, however, find the reading slow, but this is probably intrinsic to the subject matter. The notion of a Turing machine is taken as basic, and the reviewer is gratified to note that the author sharply distinguishes what a Turing machine is from how it behaves. Up to the appearance of this book hardly any two articles on Turing machines employed exactly the same concepts of a Turing machine and functions computed by such a machine. This made it necessary to check that results established for one concept held for another. Since the concepts (mentioned above) employed by the author seem highly satisfactory it is to be hoped that the tendency to introduce even more variants will be attenuated.

While the book appears suitable for the classroom, this usefulness would have been greatly enhanced by the inclusion of exercises which, in part, would indicate alternative lines of development. The reader familiar with programming digital

computers will note that construction of Turing machines is an activity similar to programming. One should be wary, however, in interpreting results concerning Turing machines as results concerning digital computers, for this implies, in particular, a willingness to endow the (idealized) digital computer with (potentially) infinite memory. For some purposes, at least, it is more appropriate to take as a mathematical model of a digital computer the notion of a finite (memory) automaton. The significance of this change in mathematical model is indicated by the fact that the analogue of the holding problem for finite automata is recursively solvable. This may be interpreted as implying the possibility of constructing a program for a Universal Turing Machine for determining whether or not an arbitrarily given program for a digital computer is free of "loops."

The chapter headings are: Computable Functions, Operations on Computable Functions, Recursive Functions, Turing Machines Self-Applied, Unsolvable Decision Problems, Combinatorial Problems, Diophantine Equations, Mathematical Logic, The Kleene Hierarchy, Computable Functionals, and The Classification of Unsolvable Decision Problems.

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50 [G, H, W, X, Z].—MATHEMATISCHES LABOR DER TECHNISCHEN HOCHSCHULE WIEN, *MTW Mitteilungen*, Vol. 5, 1958, 380 p., 24 cm. Price DM 15.

Of the 380 pages in this volume, two hundred are devoted to contributed articles, the remainder to various progress reports, reviews, news, etc. The relative proportions of these features are about the same as in earlier volumes [1]. The "literature reports" (references to new publications, arranged by subject matter) seem to have been discontinued after March, 1958.

Of the contributed articles, some are accounts of new research results, some are expository surveys, and some are reports on meetings or talks. The following are likely to be of interest to the computer mathematician. Most of the articles are in German; the titles below are translated.

W. KNÖDEL: A Transportation Problem. The usual algorithm for solving transportation problems is applied to the Austrian sugar industry, using standard punched-card machines.

W. SPINDELBERGER: Determination of complex roots of algebraic equations on small electronic computers.

W. GRÖBNER: Computation of powers of matrices. The mth power of an $n \times n$ matrix A, where m is considerably larger than n, is expressed as a polynomial $p_m(A)$ in A of degree at most n; more precisely, of degree at most d, where $d < m$ is the degree of the minimal polynomial of A. The coefficients of p_m can be found by recurrence with respect to m, if the coefficients of the minimal polynomial are known.

H. RECHBERGER and H. SEQUENZ: Basic problems in simulation of transients in reactors. A discussion of analog and mixed digital-analog systems for this problem.

K. ZUSE: "Feldrechenmaschine." This is the name given by the author to a proposed logical design for a digital computer with single commands causing arithmetic operations on "fields of data," such as vectors or matrices.

E. EGERVARY: Remarks on the transportation problem. A theorem of the author

has been used by H. W. Kuhn and M. Flood to develop an algorithm for the Assignment Problem of linear programming. In the present paper the author modifies the method of Kuhn and Flood for transportation problems with integer coefficients; these can be considered as special cases of assignment problems.

F. G. TRICOMI: Active problems in the theory of ordinary differential equations. This is a survey of the recent literature.

FRANZ L. ALT

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1. *MTAC*, v. 13, 1959, p. 67–68 (Review 10).

51 [G, S].—B. J. SEARS & M. G. RADTKE, *Algebraic Tables of Clebsch-Gordan Coefficients*, Report AECL No. 746, 1954, II + 10 p., 27 cm. Available from Scientific Document Distribution Office, Atomic Energy of Canada Limited, Chalk River, Ontario, Canada. Price $.50.

Prior to the development of quantum mechanics, the Clebsch-Gordan series played an important role in the theory of invariants of binary forms and elsewhere in group theory. In quantum mechanics, the coefficients in these series give the matrix components needed for the coupling of angular-momentum vectors. If two angular momenta having quantum numbers j_1 and j_2 and components m_1 and m_2 (all integral or half-integral) are coupled to give resultants j and m, the matrix component $(j_1\, j_2\, m_1\, m_2 \mid j\, m)$ is a Clebsch-Gordan coefficient.

The algebraic formula for these coefficients involves a complex sum; hence, to facilitate computation of the numerical values needed by theoretical physicists studying atomic and nuclear structure, algebraic tables were devised. The present volume contains six tables in which the values of j_1 are successively fixed at $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, and 3. The rows of the tables correspond to the permitted values of j, from $j_2 + j_1$ to $j_2 - j_1$ ($j_2 \geqq j_1$), while the columns correspond to the values of m_1 from $+j_1$ to $-j_1$. The entries are comparatively simple algebraic expressions in j and m.

Only the table for $j_1 = \frac{5}{2}$ represents a new calculation from the definitions. The first four tables are rearrangements of those appearing in Condon and Shortley, *Theory of Atomic Spectra* (1935), while the last was computed by Falkoff, Collady, and Sells in 1952. The rearrangement by the present authors involves expressing the entries in terms of j and m instead of j_2 and m (actually j_1 and m in previous tables where j_2 was fixed), and leads to slightly simpler algebraic expressions but no essential reduction in time required for numerical substitution.

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Bethesda, Maryland

52 [G, X].—FRANZ E. HOHN, *Elementary Matrix Algebra*, The Macmillan Company, New York, 1958, xi + 305 p., 24 cm. Price $10.00.

For several years the reviewer has taught beginning graduate courses in the computational methods of linear algebra, for which it is essential that the student have a strong knowledge of concrete linear algebra, of finite-dimensional linear analysis,

and of the analytic geometry of hyperplanes and quadric surfaces in n dimensions. Even reasonably good students who have studied Birkhoff and MacLane [1] seem astonishingly unable to use the basic theorems of linear algebra in the concrete analysis of actual computations. As an example, students need a working familiarity with the Jordan canonical form of a linear transformation and its relation to the biorthogonal system of eigenvectors and principal vectors of the transformation. Without this knowledge it is quite impossible to understand, for example, the inner working of the Liebmann process or successive overrelaxation. Another important need is a familiarity with the principal vector norms of numerical analysis—the pth-power norms for $p = 1, 2, \infty$—and the associated matrix norms.

The reviewer is often asked to recommend suitable textbooks for remedial work in this area. Such books as Perlis [2] seem a little too elementary, and skip the analysis and geometry in favor of the algebra. Perhaps the best reference has been the first chapter of Faddeeva [3], until now available only in Russian or in an informal translation. And Faddeeva, good as it is in algebra and analysis, lacks the geometric point of view and has no exercises.

The reviewer therefore welcomed the arrival of Hohn's book as a new textbook in this field, not written by a pure algebraist (the author is an associate professor of mathematics at Illinois), and directed to future applied mathematicians who are at the junior-senior level in a university. The book is now reviewed from the standpoint of numerical analysis only, although it is directed to a much wider audience.

The reviewer's conclusion is that Hohn has written an unusually good exposition of the material covered, and that the exercises are especially good. Students will find it very readable, and the author presupposes only college algebra. On the other hand, this is basically another book on algebra, with very little analysis or geometry. Moreover, the algebra, while adequate on many topics (like Hermitian forms), seems deplorably lacking in any mention of the canonical forms under similarity. For the needs of numerical analysis as the reviewer sees them, the book would have to be supplemented by other sources of linear algebra and analysis.

There are nine chapters and three appendices. Chapter 1 introduces the reader to matrix notation. Chapter 2 defines a determinant (as the sum of n! terms), and gives basic properties. Chapter 3 deals with solving linear systems and inverting matrices. Chapter 4 covers rank and equivalence of matrices, with a proof that the rank of AB cannot exceed the rank of A or B. The concept of a *number field* (subfield of the complex numbers) is used in most further work.

Linear dependence is the topic of Chapter 5, and the fundamental existence theorem for a linear algebraic system is stated here in terms of rank. Chapter 6 is devoted to vector spaces and linear transformations. We get to Sylvester's law on the nullity of AB in terms of the nullities of A and B. The Jordan canonical form is apparently nowhere even alluded to, nor is any other canonical form under similarity, and this disappoints the reviewer very much. In Chapter 7 we meet some analysis during the development of unitary and orthogonal transformations. We again meet the important existence theorem for a linear algebraic system $AX = B$, this time in terms of the orthogonality of B to the nullspace of A^* (called the *tranjugate* of A), but the nature of the general solution is not mentioned.

In Chapter 8 the characteristic equation is discussed, with the Cayley-Hamilton theorem, and we do learn about diagonalizing a Hermitian matrix. In Chapter 9 are

discussed quadratic and Hermitian forms, definiteness, and even Cochran's theorem on representing a semidefinite form as a sum of squares.

The appendices are devoted to the $\sum$ and $\prod$ notations for sums and products, to the algebra of complex numbers, and to isomorphism as a concept. There follows an excellent 7-page bibliography of books on linear algebra and its applications in pure mathematics, numerical analysis, economics, psychology, electrical engineering, chemistry, etc.

The author consistently denotes matrices and column vectors by capital letters, but frequently uses component equations. He thus happily seems to make the notation the reader's ally instead of an enemy.

One should mention that a little desk-machine numerical analysis is contained in the book. First, there is an elimination method for evaluating a determinant (called the "sweep-out" process), with a numerical example of order 3 and a reference to Dwyer [4] for more details. The other exposition is of Gaussian elimination for solving a linear algebraic system, and an abridged form thereof, with two more numerical examples of order 3. For the student interested in round-off errors, there is a reference to von Neumann and Goldstine [5], but not to its sequel [6]. There is no mention of automatic computers as a tool for solving linear systems!

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2. SAM PERLIS, *Theory of Matrices*, Addison-Wesley Press, Cambridge, Mass., 1952.
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4. PAUL S. DWYER, *Linear Computations*, John Wiley & Sons, New York, 1951.
5. JOHN VON NEUMANN & H. H. GOLDSTINE, "Numerical inverting of matrices of high order," Amer. Math. Soc., *Bull.*, vol. 53 (1947), p. 1021–1099.
6. HERMAN H. GOLDSTINE & JOHN VON NEUMANN, "Numerical inverting of matrices of high order, II," Amer. Math. Soc., *Proc.*, vol. 2 (1951), p. 188–202.

53 [H, I, X].—W. J. CUNNINGHAM, *Introduction to Nonlinear Analysis*, Electrical and Electronic Engineering Series, McGraw-Hill Book Co., Inc., New York, 1958, ix + 349 p., 23 cm. Price $9.50.

This book is the well planned and well written outgrowth of a graduate course in electrical engineering. The subject matter is the practical analysis of nonlinear ordinary differential equations. A list of the chapter headings follows.

1. Introduction
2. Numerical Methods
3. Graphical Methods
4. Equations with Known Exact Solutions
5. Analysis of Singular Points
6. Analytical Methods
7. Forced Oscillating Systems
8. Systems described by Differential-difference Equations
9. Linear Differential Equations with Varying Coefficients
10. Stability of Nonlinear Systems

The 47 pages of the second and third chapters give a short and practical account of methods for desk calculation and for graphical solution. The remaining chapters frequently refer to these methods. The book contains many excellent figures. The exposition is bolstered with carefully worked out examples for each new idea, as is appropriate for a good textbook. The homework problems are given in 16 pages at the end of the book and are arranged according to the appropriate chapter. A bibliography, with comments on relevance to the book, is given in seven pages. A five-page index completes the book.

E. I.

54 [H, S].—A. JEFFREY, "Tables of characteristic roots and functions arising in neutron transport problems," 15 p. Deposited in the UMT File.

Application of the spherical harmonic method to neutron transport problems [1], [2] requires the knowledge of the roots ν of a certain determinantal equation (equation (1)), and of the functions $G_n(\nu)$ of those roots (given in equations (2) and (3) below).

$$(1)\qquad \begin{vmatrix} 1-c & \nu & 0 & 0 & \cdots & 0 & 0 \\ \nu & 3 & 2\nu & 0 & \cdots & 0 & 0 \\ 0 & 2\nu & 5 & 3\nu & \cdots & 0 & 0 \\ 0 & 0 & 3\nu & 7 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & (N-1)\nu & 0 \\ 0 & 0 & 0 & 0 & \cdots & (2N-1) & N\nu \\ 0 & 0 & 0 & 0 & \cdots & N\nu & (2N+1) \end{vmatrix} = 0$$

$$(2)\qquad (n+1)G_{n+1}(\nu) + \frac{2n+1}{\nu} G_n(\nu) + nG_{n-1}(\nu) = 0,\ n = 1, 2, 3, \cdots, N$$

$$(3)\qquad G_0(\nu) = 1, \qquad G_1(\nu) = \frac{(c-1)}{\nu}$$

Tables have been constructed listing the positive roots ν of equation (1) for $N = 9, 11, 13$ and for c in the range 0(0.05)0.9(0.01)1.0. An abbreviated tabulation of the functions $G_n(\nu_i)$ of equations (2) and (3) is also given for argument c in the range 0(0.1)0.8(0.05)0.95, 0.99, 1.0. These 7D data have been deposited in the *MTAC* repository for Unpublished Mathematical Tables.

H. P.

1. J. C. MARK, *The Spherical Harmonics Method*, Parts I and II, Atomic Energy of Canada Limited, Reports CRT-340 and CRT-338.
2. B. DAVISON, *Neutron Transport Theory*, Oxford University Press, 1957.

55 [I].—H. E. SALZER, *Tables of Osculatory Interpolation Coefficients*, National Bureau of Standards Applied Math. Series, No. 56, U. S. Government Printing Office, Washington, D.C., 1959, xi + 25 p., 26 cm., Price $.30.

Let f(x) be a function defined over a region X, with at least $2n$ bounded derivatives in X. Let $f_i = f(x_i)$; $f_i' = [df(x)/dx]_{x=x_i}$. The well-known Lagrange-Hermite formula for osculatory interpolation has the form

$$f(x) = f(x_0 + ph) = \sum_{i=-[(n-1)/2]}^{[n/2]} \{A_i^n(p)f_i + hB_i^n(p)f_i'\} + R_{2n}$$

where

$$R_{2n} = \frac{f^{(2n)}(w)h^{2n}}{(2n)!} \prod_{j=-[(n-1)/2]}^{j=[n/2]} (p-j)^2.$$

In the above $[s]$ denotes the greatest integer in s, and w is an inner point of X.

An inspection of the form of the remainder shows that for comparable values of the derivative of order $2n$, the formula given above is more accurate than the corresponding ordinary Lagrangian formula of degree $2n$ (which involves the same amount of work). The above formula is therefore more advantageous in cases where $f(x)$ is required, and where both $f(x)$ and $f'(x)$ are known or easily obtainable at points x_i. The author tabulates the coefficients A_i and B_i over the following range: $n = 2, 3$, $p = -[(n-1)/2]$ (.01) $[(n/2)]$, exact; $n = 4, 5$, to 9D, same range and interval.

The table should prove to be very useful in application. The author gives the inexact values to within 1.5 units in the last place, but adds: "This accuracy is not absolutely guaranteed, owing to the large amount of extra work that would be necessary to check it completely." Considering the author's past performance, a table issued under his name has indeed a high probability of being accurate. It is, nevertheless, a sobering fact that rigorous methods of checking tables are nowadays considered too expensive to be worth while!

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56 [I, X].—L. Fox, *The Numerical Solution of Two-Point Boundary Problems in Ordinary Differential Equations*, Clarendon Press, Oxford, 1957, xi + 371 p., 24 cm. Price $9.60.

This volume is the first in a series of monographs on numerical analysis planned by E. T. Goodwin and the late D. R. Hartree. In it, the numerical solution of one-dimensional boundary problems by finite difference methods receives the most comprehensive treatment which has as yet appeared in print. The emphasis is on purely numerical methods of universal applicability; "semi-analytical" techniques such as the methods of Ritz and Galerkin are not treated.

The author modestly disclaims all pretensions to rigorous or highbrow mathematics. According to his own words, the book is intended for the practical computer and for the student seeking a career in computation. Yet the book can hardly be dismissed as mathematically trivial; it would take a competent analyst to prove, say, the convergence of some of the algorithms proposed in the book, or to establish rigorous error bounds. Students of numerical analysis looking for topics for Ph.D. dissertations will find ample raw material here.

The recurrent theme of the author's treatment of boundary problems is the systematic use of the *difference correction*. This is mathematically (although not numerically) equivalent to improving a crude first approximation to a solution by taking into account the error terms in the finite difference approximation to the differential operator. As the author puts his basic philosophy (p. 35): "Most writers neglect the difference correction, using more accurate formulae for the derivatives, or a small enough interval, so that the error is either negligible or kept within reason-

able bounds. In this book we shall work with a reasonably large interval and take full account of the difference correction. A large interval involves a correspondingly small number of algebraic equations and a proportionately small amount of labour: the inclusion of the difference correction yields high accuracy at this large interval." He also says in the preface: "Throughout I have taken the view that truncation errors in finite difference equations should not be tolerated." While it would be easy to take issue with these principles, if taken too literally, it cannot be denied that the author makes a strong case for a reasonable interpretation of them.

After introductory chapters of a general nature on the algebra of finite difference operators and on the solution of algebraic equations, the author considers second-order problems in Chapter 4. There follows, somewhat surprisingly, a chapter on the solution of initial-value problems by boundary-value techniques. Further chapters concern equations of higher than second order, eigenvalue problems, accuracy and precision, and miscellaneous methods. The discussion proceeds almost wholly by examples, all fully worked (on a desk computer). Many of the examples illustrate special difficulties and points of interest, such as the treatment of singularities, changes of the basic interval, and boundary conditions at infinity. Some of the famous tricks due to classical British numerical analysts, such as L. F. Richardson's deferred approach to the limit and A. C. Aitken's δ^2-method, are discussed in detail.

The author writes a terse, but at the same time fluent and highly readable English. The Clarendon Press has with this volume once more demonstrated its tradition of superb mathematical printing.

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57 [K].—FORMAN S. ACTON, *Analysis of Straight-Line Data*, John Wiley & Sons, Inc., New York, 1959, xiii + 267 p., 24 cm. Price $9.00.

As the title states, this is a book fitting a straight line to a set of data. Every conceivable set of a priori situations is assumed, and then the techniques for making the fit are fully and clearly discussed based on these assumptions. All the necessary side problems are considered and illustrated by examples.

The author's preface states very well why the book was written, and I would like to quote a few lines.

"This book was written from the conviction that we need more detailed expositions of classical and modern statistical techniques than are now possible in a general text surveying the entire field. Accordingly, I have selected the topics most pertinent to the engineer or physical scientist when he deals with data containing one or more lines entrapped within his experimental variability. He may seek to reveal the line, or he may desire to remove it in order better to examine the residual fluctuations. In either event, he must encounter the philosophy and arithmetic of the analysis of variance—a philosophy that all too often has been shoved off as a stepchild because its mathematical theory is rather tedious and pedestrian.

"In particular, I wanted to set down a strong plea for the use of these analytical techniques by the experimentalist himself, rather than by a professional statistician.

The structure of the experiment determines the proper method for analyzing its data, and the experimenter himself knows that structure best. Since his statistical background is apt to be shaky, he needs to be guided—at times in considerable detail—and he usually prefers concrete examples from which general principles may be inferred. I have therefore tried to introduce each subject with data from a real experiment—examining them at first broadly, and then again with an increase in detail that occasionally they do not deserve. When I have felt that they will not mislead, I have sometimes relied on explanations that are heuristic, but in general I have tried to emphasize the essential questions that are posed by the mathematical models our experimenter may choose in analyzing his data. It is with the influence of the model on the nature of the extractable information that I have been chiefly concerned."

The author also states that he did not intend the book for "classroom use," but rather as a detailed reference book. It is quite clear that Professor Acton has done a very excellent job and carried out fully the goal he set for himself.

After a brief introduction in Chapter 1, the author discusses in Chapters 2 and 3 the standard straight-line-fit problem in every way possible. Some needed information about the bivariate normal is presented in Chapter 4. Chapter 5 then considers the non-standard straight line fit when both variables are subject to error. Chapter 6 takes up the straight line fit in the analysis of variance and presents (for the first time in a text, to my knowledge) information of great value when "doing" an analysis of variance. Following Chapter 7, which is devoted to non-linear fit and orthogonal polynomial fit, the last three chapters round out the text by treating questions which usually come up in connection with least-squares problems, namely, transformations, rejection of extreme values, and cumulative frequency fitting.

Having this book available will really be a great boon for any statistician who is in daily contact with non-statistical scientific people. The nonstatistician would profit from exposure to it. It is an excellent book.

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58 [K, P].—STEPHEN H. CRANDALL, Editor, *Notes for the M. I. T. Special Summer Program on Random Vibration*, The Technology Press of the Massachusetts Institute of Technology, Cambridge, Massachusetts, 1958, viii + 418 p., 196 Fig., 22 cm. Price $10.00.

The first four chapters of these notes cover mechanical vibrations with deterministic excitation (S. H. Crandall), the mathematical description of random processes (W. M. Siebert), stochastic processes of mechanical origin (H. Poritsky), and statistical properties of response to random vibrations. The remaining eight chapters cover instrumentation for random vibration analysis (T. F. Rona), simulation equipment (K. J. Metzgar & D. E. Priest), and a very wide variety of discussions of special fields in which random vibration analysis is important: for example, structural damping (T. H. H. Pian), the fatigue of metals (F. A. McClintock), the response of structures to random pressures and to jet noise in particular (A. Powell),

the estimation of sound-induced missile vibrations (I. Dyer), and mechanical design for random loading (R. M. Mains).

The treatment of the basic material in the early chapters is good, but rather abbreviated, especially if the students taking the course are not already reasonably familiar with the leading concepts and the mathematical methods that have been developed for the study of random vibrations. A person without such a background would probably not get much out of this brief and rapid treatment, and one well equipped would presumably find it superfluous. The chapters on applications cover a wide field, and in each case would appear to be of interest only to specialists. It seems to me that a far sounder course would have resulted if no attempt had been made to cover detailed applications, and if instead the course had been devoted to a more careful and detailed presentation of the basic theoretical material, followed by the chapters on instrumentation and on testing devices.

C. V. L. S.

59 [L].—Louis Robin, *Fonctions Spheriques de Legendre et Fonctions Spheroidales*, Gauthier-Villars, Paris. Tome 1, 1957, xxxvi + 201 p.; Tome 2, 1958, vii + 384 p., 24 cm. Price $18.50.

The mathematical community will be grateful to Dr. Robin for providing an up-to-date and virtually encyclopedic work on Legendre functions, which is certain to take its place among the most important books on its subject, and in many respects is likely to become the standard work of reference in this field.

The book is written primarily from the point of view of applied mathematicians, having in mind the numerous classical applications of Legendre functions and spherical harmonics as well as some modern applications, such as in antenna theory. It gives a wealth of formulae and results. These are clearly stated, with the conditions of validity scrupulously described in each case. The majority of the results are proved in detail; in cases where proofs have been omitted or severely curtailed, references to the literature are included. Errors found in the literature have been corrected, so that some of the results appear here correctly for the first time, and others are new or due to the author, who has been engaged in research on Legendre functions for a number of years.

The contents of the first two volumes of this treatise will now be summarized and briefly commented upon.

Topics considered in Chapter I include the separation of the wave equation in curvilinear coordinates, Legendre's differential equation, Legendre polynomials and their properties, and Legendre functions of the second kind and their properties.

Chapter II deals with the associated Legendre functions P_n^m and Q_n^m for integral values of m and non-negative integral values of n. A special feature of this chapter is the careful consideration of P_n^m for negative integers m, including $m < -n$.

Chapter III treats spherical harmonics of (positive, negative, or zero) integral degree. Included therein is an account of Maxwell's generation of spherical harmonics and transformation formulae to new coordinate axes, as well as the addition theorem for Legendre polynomials and expansions in spherical surface harmonics. Appendices contain explicit expansions and formulae for Legendre functions and associated Legendre functions with integral m and n.

The second volume begins with Chapter IV, which occupies 211 pages. It is devoted to an investigation of the associated Legendre function $P_n^m(z)$ for general (complex) values of m, n, and z. Corresponding to such values, this function is defined by a double loop integral. The connection with the hypergeometric equation is established, and numerous hypergeometric series are given for P_n^m and Q_n^m. The analytic nature of the Legendre functions as functions of the three complex variables m, n, z is carefully described, and expansions are given also in the exceptional (logarithmic) cases as well as for the derivatives of the functions with respect to m and n. Chapter V is concerned with approximations, asymptotic expansions, and inequalities. This is the largest published collection of material of this kind, and some of it is new or has not previously appeared in a book. Chapter VI relates to expansions in Legendre polynomials, associated Legendre functions, and spherical surface harmonics. Both the (pointwise) convergence and various kinds of summability of these expansions are investigated, and some examples of such expansions are given. Expansions involving $P_{n_k}^m(\cos\theta)$ are also discussed when the n_k are roots of a transcendental equation. The Gibbs phenomenon is described.

A forthcoming third volume will contain four chapters on addition theorems, zeros, applications, and related functions, including spheroidal functions, and an appendix on numerical tables of Legendre functions.

The book is very lucidly written and can be recommended both for study and as a work of reference. References to the literature are given throughout the text at appropriate places, but there is no systematic bibliography in the two volumes under review.

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60 [P, S, T, X, Z].—L. H. HERBACH, Editor, *Symposium on Digital Computing in the Chemical and Petrochemical Industries*, New York University-International Business Machines Corporation Symposium, New York, 1958, vi + 189 p., 28 cm. Price $7.00.

This report contains a series of papers given at the Symposium on Digital Computing in the Chemical and Petrochemical Industries held at New York University in May 1958. This was the second symposium in this series. The first symposium was held in early 1957 and was devoted to the application of computers in the aircraft industry. A list of papers follows.

1. "A Numerical Solution to the Miscible Displacement Equation" by D. U. von Rosenberg.
2. "Thermodynamic Properties of Neon" by J. B. Butt, J. C. Leutwyler, B. F. Dodge & R. W. Southworth.
3. "Numerical Experiments in Chemical Reactor Design" by L. Nemerever.
4. "Design of a Distributed Control System" by O. Bilous.
5. "Supervision of an Operating Plant by a Digital Computer" by W. D. Mohr.
6. "Multicomponent Ion Exchange Column Calculations" by J. Dranoff & L. Lapidus.

7. "Multicomponent Distillation Calculation on the IBM 704" by J. Greenstadt, Y. Bard, & B. Morse.
8. "Source-Sink Method for Oil-well Fracture Problem" by M. Friedman, E. Mehr, & C. Harris.
9. "Solution of Diffusion Equations for Photochemical Initiation by Light that is Intermittent in Space" by R. M. Noyes.
10. "Molecular Orbital Calculations" by J. Auffray.
11. "The Role of the Digital Computer in the Structure Determination of Complex Organic Molecules by X-ray Analysis" by G. A. Jeffry.
12. "The Use of Digital Computers in X-ray Crystallography" by W. G. Sly.
13. "The Engineer, the Computer, and the General Purpose Program" by E. P. Bartkus & W. M. Carlson.
14. "The Use of Linear Programming Techniques in Crude Oil Production" by A. S. Lee & J. Aronofsky.
15. "Product Allocation by Linear Programming" by K. H. Shaffir & T. Zang.
16. "Library Programs for Chemical Engineering Applications" by G. Trimble.

H. P.

61 [P, X].—RUEL V. CHURCHILL, *Operational Mathematics*, McGraw-Hill Book Co., Inc., New York, 1958, ix + 337 p., 23 cm. Price $7.00.

The first edition of this book, published in 1944, was entitled *Modern Operational Mathematics in Engineering*. With the exception of the last chapter, which is concerned with Fourier transforms, both editions are devoted to an exposition of the theory of the Laplace transformation and its applications to physical problems, at about the junior-senior level of difficulty. The present edition is an enlargement by about ten per cent of the old, with many sections extensively rewritten and new problems added. Considerably rewritten are the portions of the book concerned with fundamental properties of analytic functions and with Sturm-Liouville systems. The arrangement of chapters is changed not at all, and of subject matter within chapters, very little from the earlier edition. The exposition is sound and lucid, and the book should have wide appeal.

Tables are essentially as in the earlier edition, namely, a table of Laplace transforms containing 125 entries, and a short table of finite Fourier sine and cosine transforms.

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62 [S].—J. P. SCHIFFER, *Tables of Charged Particles Penetrabilities*, ANL-5739, Argonne National Laboratory, Lemont, Ill., 1957, 59 p., 28 cm. Price $.40. Office of Technical Services, Washington 25, D. C.

Charged particles penetrabilities defined as $A_L^2(x, g) = F_L^2(x, g) + G_L^2(x, g)$ where F_L and G_L are the regular and irregular Coulomb functions, have been calculated for $L = 0$ to 4, $x \equiv \rho/2\eta = 0.05(.01)0.70$, $g \equiv \sqrt{2\rho\eta} = 1.6(.1)7.0$. The

range covered corresponds roughly for protons on nuclei to $z \simeq 15$ ($\backsim$1)75 and bombarding energies up to 0.7 of the Coulomb barrier height.

The calculations were carried out on an IBM 650, neglecting F_L^2 and using the approximate formulae of Feshbach, Shapiro and Weisskopf [1] for G_0 and G_0'.

The tables are reproduced from the machine listings in floating decimal format with all quantities given to four figures. The accuracy was checked where possible against the entries of Bloch, et al [2]; regions of agreement to better than two per cent and better than five per cent with the latter reference are indicated. Other regions are admittedly meaningless.

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1. H. FESHBACH, I. SHAPIRO & V. F. WEISSKOPF, *Tables of Penetrabilities for Charged Particle Reactions*, New York Operations Office of AEC 3077, 1953.
2. I. BLOCH, ET AL., "Coulomb functions for reactions of protons and alpha-particles with the lighter nuclei," *Revs. Mod. Phys.*, v. 23, 1951, p. 147–182.

63 [S, X].—B. NOBLE, *Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations*, Pergamon Press, Inc., New York, 1959, 246 p., 21 cm. Price $10.00.

This excellent book is a definite contribution to modern applied mathematics. It brings together, in a well-integrated manner, results which are widely dispersed in the scientific literature.

Professor Noble has isolated the fundamental idea behind the Wiener-Hopf technique. He recognizes that it deals with certain factorization problems for functions of a complex variable and that it is only indirectly related to the Wiener-Hopf integral equation with which it is usually associated.

The classical Wiener-Hopf procedure starts from the Wiener-Hopf integral equation:

$$\lambda h(x) + f(x) = \int_0^\infty k(x-t)h(t)\,dt\,, \qquad 0 < x < \infty\,. \tag{1}$$

In the applications which are considered in Noble's book, λ vanishes. Define $h(x)$ and $f(x)$ to be 0 for negative x, and let $e(x)$ be the value of the integral in (1) for negative x. Introduction of Fourier transforms reduces equation (1) to:

$$F_+(\alpha) + E_-(\alpha) = K(\alpha)\,H_+(\alpha), \tag{2}$$

where F_+ and H_+ are analytic in an upper half plane, E_- is analytic in a lower half plane, K is analytic in a strip. $K(\alpha)$ can be factored as K_+/K_- and F_+K_- can be split as $C_+ + C_-$. This yields

$$E_-(\alpha)K_-(\alpha) + C_-(\alpha) = H_+(\alpha)K_+(\alpha) - C_+(\alpha), \tag{3}$$

where E_- and H_+ are unknown. The remarkable fact is that both can be determined from the single equation (3). The left and right sides of (3) usually have a common strip of analyticity, so that either side is the analytic continuation of the other. Both sides are therefore equal to the same entire function $M(\alpha)$, which often turns out to be a constant or a polynomial. $M(\alpha)$ can be determined from the asymptotic be-

havior of the functions appearing in equation (3). Some of these asymptotic behaviors are postulated on physical grounds, the others being found from equation (1).

Once $M(\alpha)$ has been determined, H_+ and E_- are computed from (3), and inversion gives $h(x)$.

A typical physical application occurs in the diffraction of waves by a half-plane. The boundary value problem can be reduced to an integral equation of type (1), where $k(x - t)$ is the Green's function for the partial differential equation, $h(x)$ is the current on the screen, and $f(x)$ is the incoming wave on the screen.

Noble points out that the functional equation (2) can be obtained directly from the boundary-value problem without reference to the integral equation. The advantages of this point of view are the following:

a. $K(\alpha)$ can be obtained with ease by transforming the partial differential equation directly without introducing the Green's function k explicitly.

b. We are led to possible further applications of the Wiener-Hopf technique to problems which cannot be formulated as an equation of type (1). The generalized two-part boundary-value problems of S. Karp fall in this category. The Wiener-Hopf technique can be extended to cope with these problems, the appropriate transform being a Mellin, Hankel, or Kantorovich-Lebedev transform rather than the ordinary Fourier transform used for problem (1).

There are many other interesting features of Noble's book. A comparison of the Wiener-Hopf technique with Titchmarsch's dual integral equation approach is given. One also finds a careful discussion of the relationship between the Wiener-Hopf problem and the Hilbert problem. An extensive collection of problems (taken from acoustics, electromagnetic theory, fluid flow, elasticity, and neutron diffusion) is worked out by the author. He has also included a wealth of non-trivial exercises, many of which are likely starting points for further research.

There are a number of topics omitted by Professor Noble, which I would have liked to see included. These omissions do not detract appreciably from the over-all excellence of the volume, but they might easily be remedied in the next edition:

a. The work on approximate methods in connection with Wiener-Hopf problems. The powerful techniques developed by Harold Levine receive only cursory mention.

b. The attempt to develop a two-dimensional Wiener-Hopf technique. J. Radlow has made a start toward the solution of the quarter-plane problem. This leads to difficult questions in the theory of functions of two complex variables.

c. There is a fairly extensive literature in probability theory on the Wiener-Hopf method. The interest is with solutions of integral equations of type (1) which are required to be cumulative probability distributions. Significant results have been obtained in this direction by F. Pollaczek, W. L. Smith, and F. Spitzer.

In conclusion, Professor Noble is to be thanked for his valuable analysis of an important tool of mathematical physics.

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64 [W, X].—K. J. ARROW, L. HURWICZ, & H. UZAWA, *Studies in Linear and Non-Linear Programming*, Stanford Mathematical Studies in the Social Sciences, II,

Stanford University Press, Stanford, California, 1958, iv + 229 p., 25 cm. Price $7.50.

The general mathematical programming problem can be defined as: Determine the n-vector x which maximizes $f(x)$ subject to the conditions

$$g_i(x) \geqq 0, \qquad i = 1, 2, \ldots m$$
$$x \geqq 0.$$

Linear programming is the special case where $f(x)$ and the constraints $g_i(x)$ are linear.

Since the development of the Simplex Method, linear programming has had many applications to management problems. Also, as a representation of the economic problem of optimum allocation of scarce resources, linear programming (with its duality theorem) has become an important concept in economic analysis.

Refinements to linear programming have been extensive in recent years, but developments in the more general aspects of mathematical programming have not been nearly as rapid. Computational procedures have been developed for the case where the functional $f(x)$ is quadratic and where $f(x)$ is convex and separable for each x_j. All of these procedures have used the Simplex Method, or some variant, as the basic computational technique. Of more general nature is the basic paper by H. Kuhn and A. W. Tucker entitled "Non-linear Programming," which appeared in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics*, 1951, p. 481–492. In this paper, Kuhn and Tucker give conditions under which the mathematical programming problem is equivalent to finding the saddle point of the Lagrangian function, $\varphi(x, u) = f(x) + u \cdot g(x)$.

The Kuhn-Tucker theorem on non-linear programming provides the basis of this book, which is in three parts. Part I, entitled Existence Theorems, is mainly concerned with extensions of the Kuhn-Tucker theorem. Part II, entitled Gradient Method, is devoted to extensive studies of the gradient method applied to the saddle point of the Lagrangian function. Of special interest is the development of a modified gradient method which converges for all concave programming problems, (that is, where $f(x)$ and $g(x)$ are concave functions of x) without restriction. Examples of this modified gradient method applied to two small linear programming problems are presented in detail. Part III, entitled Methods of Linear and Quadratic Programming, discusses the solution of three specific problems whose special structures are used to develop methods of solution. Also, a method for solving linear programs based on a theorem on extreme points of convex polyhedral cones is presented.

This book, which is a collection of related papers published for the first time, is the second in the Stanford Mathematical Studies in the Social Sciences series. Like the first, *Studies in the Mathematical Theory of Inventory and Production*, by Arrow, Karlin, and Scarf, it is a valuable addition to the literature. It should be of considerable interest to mathematical economists and management scientists concerned with generalizations of linear programming.

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65 [W, X].—A. W. Tucker & R. D. Luce, *Contributions to the Theory of Games*, Annals of Mathematics Studies, v. 4, Princeton University Press, New Jersey, 1959, 453 p., 24 cm. Price $6.00.

This is the fourth in a series of books presenting papers relating to the theory of games; the editors claim that it is the last. The whole series has been a most valuable collection of papers; if nothing else, they indicate the mathematical, and hence the numerical, difficulty which is to be encountered in attempts to treat competitive situations quantitatively.

The present volume is perhaps further removed from computational method than any of the earlier ones. It still contributes to the description of the quantitative difficulties occurring in the description of competitive situations.

One of the most valuable contributions of this volume is a bibliography by Dorothea M. Thompson and Gerald L. Thompson; this is a list of about a thousand of the most readily available contributions to game theory.

A second valuable contribution is the introduction, presumably written by the editors, outlining what is in the volume and what the general significance of each paper is. This introduction constitutes what would be the most valuable review possible for this volume, but it is unfortunately too long to reproduce here.

A third interesting item is a translation into English by Mrs. Sonya Bargmann of von Neumann's famous early paper "Zur Theorie der Gesellschaftsspiele," which appeared in 1928.

While the present volume has only these fairly remote offerings to make to computation, the fact that so much present day computing is concerned with economic problems makes the material worth noting here. In addition, the reviewer would like to add his applause to that already bestowed on the editors of this series of volumes for their contributions in publishing research works in a single field grouped into a single volume with a good editorial introduction to guide the reader. The time the editors have expended on this task is certainly great, but to the reader the contribution is most valuable. An expansion of this idea would be most valuable if the editorial time could be made available. Specifically, it would seem appropriate to have journal editors relay some of the information they receive in the form of referees' reports to aid the cursory readers of their journals.

C. B. T.

66 [X].—Glenn James & Robert C. James, Editors, *Mathematics Dictionary*, 2nd Edition, D. Van Nostrand, Princeton, 1959, 546 p., 24 cm. Price $15.00.

The new edition of this well known reference is about 25 percent longer than the 1949 edition. The new material includes terms from "modern algebra, number theory, topology, vector spaces, the theory of games and linear and dynamic programming, numerical analysis, and computing machines." Entirely new are the four foreign language dictionaries of mathematical terms in the appendix: French—19 pages, German—16 pages, Russian—22 pages, and Spanish—15 pages.

The dictionary contains a wealth of information and makes interesting browsing. While the book is a "must" for mathematical libraries, it is not difficult to find many minor defects. Since the editors, in their preface, invite comments, here are some of mine.

Sins of omission and commission: Where are "Class Number," "Continuum Hypothesis", "Diophantine Approximation", "Dirichlet Series", and "Distribution, (L. Schwartz)"? On the other hand must we have "Commercial Bank", "Horsepower", "Mariner's Compass", etc.? Sins of misplaced emphasis: After "Möbius function" one would expect something about the Möbius inversion formula, but instead we find the more sophisticated, but less fundamental, relation to the Riemann Hypothesis. Improper definitions: "Unique Factorization Theorem" and "Factorial Series" are not properly defined. Improper spelling: "Fresnal", "Marriot". While "Tchebycheff" and even "Tschebyscheff" are not "wrong," the more modern "Chebyshev" could at least be given as a cross-reference. Typography: "Joukowski Transformation" is listed as a subheading of "Jordan." Non sequitur: "Fibonacci Sequence" has a mysterious reference to "Farey Sequence", but this should cause the reader little inconvenience since it is only one page away.

The appendices contain elementary tables of logarithms, trigonometric functions, compound interest, etc. There is also a table of "Denominate Numbers" (2 pints = 1 quart, 7 days = 1 week, 20 grains = 1 scruple, 2352 pounds = 1 Cornish mining ton, 24 sheets = 1 quire, 500 sheets = 1 long ream, etc.). British readers susceptible to acute nostalgia should be forewarned of the item which states that the British monetary pound is normally equal to 5 dollars. The editors seem especially fond of the fact that a gallon equals 231 cubic inches, since this is repeated three times. With due respect to them, could not the space be better used in the tables of integrals which follows? The type size there is almost unbelievably small.

The foreign-language dictionaries and the hundreds of new terms such as "Haar measure", "Tychonoff space", etc., add considerably to the value of this useful reference book.

D. S.

67 [Z].—A. D. Booth & K. H. V. Booth, *Automatic Digital Calculators*, Second Edition, Academic Press Inc., New York, 1956, 261 p., 21 cm. Price $6.00.

This book is a non-mathematical introduction to the principles and applications of computers employing tubes and other electronic devices. It is a very general coverage of the entire digital computing field.

The first three chapters of the book are devoted to the evolution of calculating machines to the modern general-purpose electronic digital computer.

The next four chapters discuss the overall design of a computing system: the memory, control, arithmetic unit, and input-output units. Each unit is described in terms of serial, parallel, and decimal operation, along with the peculiarities of each type of operation and number system being used.

Four chapters are devoted to a general discussion of circuits and hardware which make up the logic of the different systems. The different types of storage, such as magnetic core, magnetic drum, magnetic disc, magnetic wire, mercury delay lines, magnetostrictive delay lines, and other types of storage, are described in terms of general operating principles.

The remaining five chapters are devoted to coding for a digital computer and digital computer applications. The definition of a code and discussion of its form and contents are described.

The book gives the reader a very good introduction to the general principles of coding, operation and application of high-speed electronic computers and computer components. Some previous background in mathematics, or a previous knowledge of computer circuitry, is required for a thorough understanding of the contents. In clear and concise terms, the authors cover the subjects in seventeen well written chapters. The book is written primarily for new workers in the field. Engineers in other fields who wish to become acquainted with the subject of computers will find the text both interesting and informative. The book lays a firm foundation for subsequent reading in the digital computer field. It is not intended, however, that the book be of use to engineers who are well informed in computing techniques and equipment. The excellent illustrations of circuitry serve well as an aid to comprehension.

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68 (Z).—WATTS S. HUMPHREY, JR., *Switching Circuits with Computer Applications*, McGraw-Hill Book Company, Inc., New York, 1958, viii + 264 p., 23 cm. Price $8.50.

After an introductory chapter mostly devoted to the binary number system then follows a fairly long chapter on Boolean algebra: the presentation is intuitive and practical, the basic ideas and results, for example, being set forth by means of Venn diagrams. This is immediately applied in Chapter 3 to the study of relay contact networks. Chapter 4 is devoted to codes; included are discussions of the various kinds of undetected errors using binary-coded decimal and excess-three codes, of modulo-nine checking, of parity checking, and of error-correcting codes. Chapter 5 is on minimizing aids, and is done almost entirely in terms of the Karnaugh map. Chapter 6, on circuit logic, covers two- and three-level diode logic and transistor logic. Chapter 7, on Boolean matrices, deals with connection matrices of networks, their modification and manipulation. This lays the foundation for the treatment, in Chapter 8, of bilateral networks, both symmetrical and antisymmetrical, and their reduction. Chapter 9, on cascaded networks, deals with complex networks built up by cascading simpler ones, and considers clocked and unclocked pulse logic and level logic. Finally, Chapter 10 is devoted to sequential circuits, defined as circuits containing memory (or time-delay) elements.

The discussion throughout is practical and didactic rather than theoretical. Many worked examples are included in the text; these are very instructive, in most cases are easy to follow, and supplement and elucidate the occasionally rather sketchy presentation of ideas. This should make the book particularly valuable for self-study, though it is also very suitable as a textbook. The emphasis throughout is on practical methods of minimizing switching networks, and no single method or procedure is presented as a cure-all. The very sensible conclusion of the introduction to the chapter on Boolean matrices could well be taken as a maxim: "It is, however, no different from the other switching-circuit design techniques: it is an aid to the designer and does not replace skill and experience."

C. V. L. S.

NOTES

Announcement of New Title for *Mathematical Tables and Other Aids to Computation*

In order to reflect the broadened scope of *Mathematical Tables and other Aids to Computation*, and to satisfy an increasing need in this country for a journal devoted to numerical analysis and computation, the members of the editorial committee of MTAC voted unanimously at their last meeting to recommend a change in the name of the journal. *Mathematics of Computation* has been selected as the new title. This change has been approved by the executive committee of the Division of Mathematics, National Academy of Sciences-National Research Council, and will become effective with the January 1960 issue.

The new title for *Mathematical Tables and other Aids to Computation* in no way represents a diminished interest by the editorial committee in mathematical tables, which will continue to be a subject of prime emphasis, as in the past. It recognizes, however, an increased interest in other areas in the field of *Mathematics of Computation*, which have grown in importance and in which rapid advances are being made in the present era of technological progress. The subscribers and readers of *Mathematical Tables and other Aids to Computation* may expect future issues of the journal to be a continuation of MTAC, with similar style, format and character of contents, but with increased emphasis on modern advances in the theory and application of computational methods. It will appear under the abridged title, ***Mathematics of Computation.***

H. P.

New Journal

Information and Control is a journal devoted to the publication of original papers in the theories of communication, computers, and automatic control. Papers on experimental results in this field are also accepted. It is published by Academic Press Incorporated, New York, New York. Editors: Leon Brillouin, Colin Cherry, and Peter Elias. Volume I covers the years 1957–58.

E. I.

Conference on Communication and Scientific Information

The October 1958 issue, Volume 2, Number 4 of the IBM *Journal of Research and Development* is devoted to ten papers (plus a related contributed article) presented at the Conference on Communication and Scientific Information held on the occasion of the dedication of the IBM Research Laboratory, San Jose, California, May 26, 27, 1958. New developments are presented in the use of machines for efficient information processing and problem solving.

E. I.

Funds for the Publication of Mathematical Tables

Funds are available to subsidize the publication of suitable mathematical tables through the Division of Mathematics of the National Research Council of the National Academy of Sciences. Inquires should be sent to:

Professor George E. Forsythe
Chairman, Committee on Revolving Fund for Publication
of Mathematical Tables
Mathematics Department,
Stanford University,
Stanford, California

E. I.

CORRIGENDA

A. M. Ostrowski, "On Gauss' Speeding Up Device in the Theory of Single Step Iteration," *MTAC*, v. 12, 1958, p. 116–132.

p. 124, formula (36), add to the last right-side term the factor ξ.

p. 124, immediately preceding formula (36), add, "for a sufficiently small positive ξ."

p. 124, end of section 21, add, "replacing $a_1 + a_2$ by $a_1 + a_2 + \xi(n-2)$."

Walter Gautschi

Notes, *MTAC*, v. 12, 1958, p. 319, item (4) announces the publication of the monograph series *Acta Polytechnica*, Applied Mathematics and Computing Machinery Series. The more accurate title of this series is *Acta Polytechnica Scandinavica*, Mathematics and Computing Machinery Series. This joint Scandinavian series is published under the auspices of the Scandinavian Council for Applied Research.

H. P.

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